9.9 Special Matrices:

1. Transpose of a Matrix

If A = $[a_{ij}]$ is mxn matrix, then the matrix of order n x m obtained by interchanging the rows and columns of A is called the transpose of A. It is denoted A^t or A'.

Example if $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, then $A^t = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$

2. Symmetric Matrix:

A square matrix A is called symmetric if $A = A^t$

for example if

$$A = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}, \text{ then } A^t = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} = A$$

Thus A is symmetric

3. Skew Symmetric:

A square matrix A is called skew symmetric if $A = -A^t$

for example if $B = \begin{bmatrix} 0 & -4 & 1 \\ 4 & 0 & -3 \\ -1 & 3 & 0 \end{bmatrix}$,then

$$B^t = \begin{bmatrix} 0 & 4 & -1 \\ -4 & 0 & 3 \\ 1 & -3 & 0 \end{bmatrix} = (-1)\begin{bmatrix} 0 & -4 & 1 \\ 4 & 0 & -3 \\ -1 & 3 & 0 \end{bmatrix}$$

$$B^t = -B$$

Thus matrix B is skew symmetric.

4. Singular and Non-singular Matrices:

A square matrix A is called singular if $|A| = 0$ and is non-singular if $|A| \neq 0$, for example if

$$A = \begin{bmatrix} 3 & 2 \\ 9 & 6 \end{bmatrix}, \text{ then } |A| = 0, \text{ Hence A is singular}$$

and if $A = \begin{bmatrix} 3 & 1 & 6 \\ -1 & 3 & 2 \\ 1 & 0 & 0 \end{bmatrix}$, then $|A| \neq 0$,

Hence A is non-singular.

Example: Find k If $A = \begin{bmatrix} k-2 & 1 \\ 5 & k+2 \end{bmatrix}$ is singular

Solution: Since A is singular so $\begin{vmatrix} k-2 & 1 \\ 5 & k+2 \end{vmatrix} = 0$

$$(k-2)(k+2) - 5 = 0$$

$$k^2 - 4 - 5 = 0$$

$$k^2 - 9 = 0 \Rightarrow \quad K = \pm 3$$

5. Adjoint of a Matrix:

Let $A = (a_{ij})$ be a square matrix of order n x n and (c_{ij}) is a matrix obtained by replacing each element a_{ij} by its corresponding cofactor c_{ij} then $(c_{ij})^t$ is called the adjoint of A. It is written as adj. A.

For example , if

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

Cofactor of A are:

$A_{11} = 5$, $A_{12} = -2$, $A_{13} = +1$

$A_{21} = -1$, $A_{22} = 2$, $A_{23} = -1$

$A_{31} = 3$, $A_{32} = -2$, $A_{33} = 3$

Matrix of cofactors is

$$C = \begin{bmatrix} 5 & -2 & +1 \\ -1 & 2 & -1 \\ 3 & -2 & 3 \end{bmatrix}$$

$$C^t = \begin{bmatrix} 5 & -1 & 3 \\ -2 & 2 & -2 \\ +1 & -1 & 3 \end{bmatrix}$$

Hence adj A = C^t $= \begin{bmatrix} 5 & -1 & 3 \\ -2 & 2 & -2 \\ +1 & -1 & 3 \end{bmatrix}$

Note: Adjoint of a 2×2 Matrix:

The adjoint of matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is denoted by adjA is defined as

$$\text{adj}A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

6. Inverse of a Matrix:

If A is a non-singular square matrix , then $A^{-1} = \dfrac{\text{adj } A}{|A|}$

For example if matrix $A = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$

Then adj A $= \begin{bmatrix} 2 & -4 \\ -1 & 3 \end{bmatrix}$

$$|A| = \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = 6 - 4 = 2$$

Hence $A^{-1} = \dfrac{\text{adj } A}{|A|} = \dfrac{1}{2}\begin{bmatrix} 2 & -4 \\ -1 & 3 \end{bmatrix}$

Alternately:

For a non singular matrix A of order (n x n) if there exist another matrix B of order (n x n) Such that their product is the identity matrix I of order (n x n) i.e., AB = BA = I

Then B is said to be the inverse (or reciprocal) of A and is written as $B = A^{-1}$

Example 9: If $A = \begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix}$ then show that

AB = BA = I and therefore, $B = A^{-1}$

Solution:

$$AB = \begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix}\begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and $\quad BA = \begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Hence AB = BA = I

and $\quad$ therefore $B = A^{-1} = \begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix}$

Example 10: Find the inverse, if it exists, of the matrix.

$$A = \begin{bmatrix} 0 & -2 & -3 \\ 1 & 3 & 3 \\ -1 & -2 & -2 \end{bmatrix}$$

Solution:

$|A| = 0 + 2(-2 + 3) - 3(-2 + 3) = 2 - 3$

$|A| = -1$, Hence solution exists.

Cofactor of A are:

$A_{11} = 0$, $\quad A_{12} = 1$, $\quad A_{13} = 1$

$A_{21} = 2$, $\quad A_{22} = -3$, $\quad A_{23} = 2$

$A_{31} = 3$, $\quad A_{32} = -3$, $\quad A_{33} = 2$

Matrix of transpose of the cofactors is

$$\text{adj } A = C' = \begin{bmatrix} 0 & 2 & 3 \\ -1 & -3 & -3 \\ 1 & 2 & 2 \end{bmatrix}$$

So

$$A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{-1} \begin{bmatrix} 0 & 2 & 3 \\ -1 & -3 & -3 \\ 1 & 2 & 2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 0 & -2 & -3 \\ 1 & 3 & 3 \\ -1 & -2 & -2 \end{bmatrix}$$

9.11 Solution of Linear Equations by Matrices:

Consider the linear system:

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \text{------} + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \text{------} + a_{2n}x_n &= b_2 \\ | \qquad\quad | \qquad\qquad\qquad | \qquad & \\ a_{n1}x_1 + a_{n2}x_2 + \text{------} + a_{nn}x_n &= b_n \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots \quad (1)$$

It can be written as the matrix equation

$$\begin{bmatrix} a_{11} & a_{12} & \text{------} & a_{1n} \\ a_{21} & a_{22} & \text{------} & a_{2n} \\ | & | & & | \\ a_{n1} & a_{n2} & \text{------} & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ | \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ | \\ b_n \end{bmatrix}$$

Let $A = \begin{bmatrix} a_{11} & a_{12} & \text{------} & a_{1n} \\ a_{21} & a_{22} & \text{------} & a_{2n} \\ | & | & & | \\ a_{n1} & a_{n2} & \text{------} & a_{nn} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ | \\ x_n \end{bmatrix}$

$B = \begin{bmatrix} b_1 \\ b_2 \\ | \\ b_n \end{bmatrix}$

Then latter equation can be written as,

$$AX = B$$

If $B \neq 0$, then (1) is called non-homogenous system of linear equations and if $B = 0$, it is called a system of homogenous linear equations.

If now $B \neq 0$ and A is non-singular then A^{-1} exists.

Multiply both sides of $AX = B$ on the left by A^{-1}, we get

$$A^{-1}(AX) = A^{-1}B$$
$$(A^{-1}A)X = A^{-1}B$$
$$1X = A^{-1}B$$
$$\text{Or} \qquad X = A^{-1}B$$

Where $A^{-1}B$ is an n x 1 column matrix. Since X and $A^{-1}B$ are equal, each element in X is equal to the corresponding element in $A^{-1}B$. These elements of X constitute the solution of the given linear equations.

If A is a singular matrix, then of course it has no inverse, and either the system has no solution or the solution is not unique.

Example 11: Use matrices to find the solution set of

$$x + y - 2z = 3$$
$$3x - y + z = 5$$
$$3x + 3y - 6z = 9$$

Solution:

Let $A = \begin{bmatrix} 1 & 1 & -2 \\ 3 & -1 & 1 \\ 3 & 3 & -6 \end{bmatrix}$

Since $|A| = 3 + 21 - 24 = 0$

Hence the solution of the given linear equations does not exists.

Example 12: Use matrices to find the solution set of

$$4x + 8y + z = -6$$
$$2x - 3y + 2z = 0$$
$$x + 7y - 3z = -8$$

Solution:

Let $A = \begin{bmatrix} 4 & 8 & 1 \\ 2 & -3 & 2 \\ 1 & 7 & -3 \end{bmatrix}$

Since $|A| = -32 + 48 + 17 = 61$

So A^{-1} exists.

$$A^{-1} = \frac{1}{|A|} \text{adj } A$$
$$= \frac{1}{61}\begin{bmatrix} -5 & 31 & 19 \\ 8 & -13 & -16 \\ 17 & -20 & -28 \end{bmatrix}$$

Now since,

$X = A^{-1} B$, we have

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{61}\begin{bmatrix} -5 & 31 & 19 \\ 8 & -13 & -16 \\ 17 & -20 & -28 \end{bmatrix}\begin{bmatrix} -6 \\ 0 \\ -8 \end{bmatrix}$$

$$= \frac{1}{61}\begin{bmatrix} 30+152 \\ -48+48 \\ -102+224 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$$

Hence Solution set: {(x, y, z)} = {(–2, 0, 2)}

Exercise 9.3

Q.1 Which of the following matrices are singular or non-singular.

(i) $\begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & -2 \\ 0 & 1 & -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & -1 \\ -3 & 4 & 5 \\ -4 & 2 & 6 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 1 & -2 \\ 3 & -1 & 1 \\ 3 & 3 & -6 \end{bmatrix}$

Q.2 Which of the following matrices are symmetric and skew-symmetric

(i) $\begin{bmatrix} 2 & 6 & 7 \\ 6 & -2 & 3 \\ 7 & 3 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & 3 & -5 \\ -3 & 0 & 6 \\ 5 & -6 & 0 \end{bmatrix}$ (iii) $\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$

Q.3 Find K such that the following matrices are singular

(i) $\begin{vmatrix} K & 6 \\ 4 & 3 \end{vmatrix}$ (ii) $\begin{vmatrix} 1 & 2 & -1 \\ -3 & 4 & K \\ -4 & 2 & 6 \end{vmatrix}$ (iii) $\begin{vmatrix} 1 & 1 & -2 \\ 3 & -1 & 1 \\ k & 3 & -6 \end{vmatrix}$

Q.4 Find the inverse if it exists, of the following matrices

(i) $\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & -2 & -3 \\ 1 & 3 & 3 \\ -1 & -2 & -2 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \\ 0 & 2 & 2 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 2 & -1 \\ -3 & 4 & 5 \\ -4 & 2 & 6 \end{bmatrix}$

Q.5 Find the solution set of the following system by means of matrices:

(i) $2x - 3y = -1$, $x + 4y = 5$

(ii) $x + y = 2$, $2x - z = 1$, $2y - 3z = -1$

(iii) $x - 2y + z = -1$, $3x + y - 2z = 4$, $y - z = 1$

(iv) $-4x + 2y - 9z = 2$, $3x + 4y + z = 5$, $x - 3y + 2z = 8$

(v) $x + y - 2z = 3$, $3x - y + z = 0$, $3x + 3y - 6z = 8$

Answers 9.3

Q.1 (i) Non-singular (ii) Singular
(iii) Singular

Q.2 (i) Symmetric (ii) Skew-symmetric
(iii) Symmetric

Q.3 (i) 8 (ii) 5 (iii) 3

Q.4 (i) $\begin{bmatrix} \frac{1}{7} & \frac{3}{7} \\ \frac{2}{7} & \frac{1}{-7} \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & 2 & 3 \\ -1 & -3 & -3 \\ 1 & 2 & 2 \end{bmatrix}$ (iii) $\begin{bmatrix} \frac{4}{5} & -\frac{1}{5} & -\frac{4}{5} \\ -\frac{1}{5} & -\frac{1}{5} & \frac{7}{10} \\ \frac{1}{5} & \frac{1}{5} & -\frac{1}{5} \end{bmatrix}$

(iv) A^{-1} does not exist.

Q.5 (i) {(1, 1)} (ii) {(1, 1, 1)} (iii) {(1, 1, 0)}
(iv) {(7, -3, -4)} (v) no solution

Summary

1. If $A = [a_{ij}]$, $A = [b_{ij}]$ of order m x n. Then $A + B = [a_{ij} + b_{ij}]$ is also m x n order.
2. The product AB of two matrices A and B is conformable for multiplication if No of columns in A = No. of rows in B.
3. If $A = [a_{ij}]$ is m x n matrix, then the n x m matrix obtained by interchanging the rows and columns of A is called the transpose of A. It is denoted by A^t.
4. Symmetric Matrix:
   A square matrix A is symmetric if $A^t = A$.
5. If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ Then,

   (i) $\text{adj } A = \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix}$, a_{ij} are the co-factor elements.

   And inverse of A is:

(ii) $A^{-1} = \dfrac{\text{adj } A}{|A|}$

6. A square matrix A is singular if $|A| = 0$.