Q.6 $\begin{bmatrix} 6 & 17 \\ 8 & 9 \end{bmatrix}$

9.5 Determinants:

The Determinant of a Matrix:

The determinant of a matrix is a scalar (number), obtained from the elements of a matrix by specified, operations, which is characteristic of the matrix. The determinants are defined only for square matrices. It is denoted by det A or |A| for a square matrix A.

The determinant of the (2 x 2) matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is given by det A = |A| = $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

$$= a_{11}\, a_{22} - a_{12}\, a_{21}$$

Example 3: If A = $\begin{bmatrix} 3 & 1 \\ -2 & 3 \end{bmatrix}$ find |A|

Solution:

$$|A| = \begin{vmatrix} 3 & 1 \\ -2 & 3 \end{vmatrix} = 9 - (-2) = 9 + 2 = 11$$

The determinant of the (3 x 3) matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \text{ denoted by } |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

is given as, det A = |A|

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Note: Each determinant in the sum (In the R.H.S) is the determinant of a submatrix of A obtained by deleting a particular row and column of A.

These determinants are called minors. We take the sign + or −, according to $(-1)^{i+j} a_{ij}$

Where i and j represent row and column.

9.6 Minor and Cofactor of Element:

The minor M_{ij} of the element a_{ij} in a given determinant is the determinant of order $(n - 1 \times n - 1)$ obtained by deleting the ith row and jth column of A_{nxn}.

For example in the determinant

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \dots\dots\dots\dots\dots\dots\dots (1)$$

The minor of the element a_{11} is $M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

The minor of the element a_{12} is $M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

The minor of the element a_{13} is $M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$ and so on.

The scalars $C_{ij} = (-1)^{i+j} M_{ij}$ are called the cofactor of the element a_{ij} of the matrix A.

Note: The value of the determinant in equation (1) can also be found by its minor elements or cofactors, as

$a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$ Or $a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$

Hence the det A is the sum of the elements of any row or column multiplied by their corresponding cofactors.

The value of the determinant can be found by expanding it from any row or column.

Example 4: If $A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & -2 \\ 1 & 3 & 4 \end{bmatrix}$

find det A by expansion about (a) the first row (b) the first column.

Solution (a)

$$|A| = \begin{vmatrix} 3 & 2 & 1 \\ 0 & 1 & -2 \\ 1 & 3 & 4 \end{vmatrix}$$

$$= 3\begin{vmatrix}1 & -2\\3 & 4\end{vmatrix} - 2\begin{vmatrix}0 & -2\\1 & 4\end{vmatrix} + 1\begin{vmatrix}0 & 1\\1 & 3\end{vmatrix}$$

$= 3(4 + 6) - 2(0 + 2) + 1\ (0 - 1)$

$= 30 - 4 - 1$

$|A| \quad = 25$

(b) $|A| \quad = 3\begin{vmatrix}1 & -2\\3 & 4\end{vmatrix} - 0\begin{vmatrix}2 & 1\\3 & 4\end{vmatrix} + 1\begin{vmatrix}0 & 1\\1 & 3\end{vmatrix}$

$= 3\ (4 + 6) + 1\ (-4 -1)$

$= 30 - 5$

$|A| \quad = 25$

9.7 Properties of the Determinant:

The following properties of determinants are frequently useful in their evaluation:

1. Interchanging the corresponding rows and columns of a determinant does not change its value (i.e., |A| = |A'|). For example, consider a determinant

$$|A| = \begin{vmatrix}a_1 & b_1 & c_1\\a_2 & b_2 & c_2\\a_3 & b_3 & c_3\end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2) \ldots \ (2)$

Now again consider

$$|B| = \begin{vmatrix}a_1 & a_2 & a_3\\b_1 & b_2 & b_3\\c_1 & c_2 & c_3\end{vmatrix}$$

Expand it by first column

$|B| \quad = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$

which is same as equation (2)

so $\quad |B| = \begin{vmatrix}a_1 & b_1 & c_1\\a_2 & b_2 & c_2\\a_3 & b_3 & c_3\end{vmatrix}$

or $\quad |B| = |A|$

2. If two rows or two columns of a determinant are interchanged, the sign of the determinant is changed but its absolute value is unchanged.

For example if

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Consider the determinant,

$$|B| = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

expand by second row,

$$|B| = -a_1(b_2c_3 - b_3c_2) + b_1(a_2c_3 - a_3c_2) - c_1(a_2b_3 - a_3b_2)$$

$$= -(a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2))$$

The term in the bracket is same as the equation (2)

So $$|B| = -\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Or $|B| = -|A|$

3. If every element of a row or column of a determinant is zero, the value of the determinant is zero. For example

$$|A| = \begin{vmatrix} 0 & 0 & 0 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= 0(b_2c_3 - b_3c_2) - 0(a_2c_3 - a_3c_2) + 0(a_2b_3 - a_3b_3)$$

$$|A| = 0$$

4. If two rows or columns of a determinant are identical, the value of the determinant is zero. For example, if

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= a_1(b_1c_3 - b_3c_1) - b_1(a_1c_3 - a_3c_1) + c_1(a_1b_3 - a_3b_1)$$

$$= a_1b_1c_3 - a_1b_3c_1 - a_1b_1c_3 + a_3b_1c_1 + a_1b_3c_1 - a_3b_1c_1$$

$$|A| = 0$$

5. If every element of a row or column of a determinant is multiplied by the same constant K, the value of the determinant is multiplied by that constant. For example if,

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Consider a determinant, $|B| = \begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$|B| = ka_1(b_2c_3 - b_3c_3) - kb_1(a_2c_3 - a_3c_2) + kc_1(a_2b_3 - a_3b_2)$

$= k(a_1(b_2c_3 - b_3c_3) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2))$

So $|B| = k \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Or $|B| = K|A|$

6. The value of a determinant is not changed if each element of any row or of any column is added to (or subtracted from) a constant multiple of the corresponding element of another row or column. For example, if

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Consider a matrix,

$$|B| = \begin{vmatrix} a_1 + ka_2 & b_1 + kb_2 & c_1 + kc_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$= (a_1+ka_2)(b_2c_3-b_3c_2)-(b_1+kb_2)(a_2c_3-a_3c_2)+(c_1+kc_2)(a_2b_3-a_3b_2)$

$= [a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)]$

$= [ka_2(b_2c_3 - b_3c_2) - kb_2(a_2c_3 - a_3c_2) + kc_2(a_2b_3 - a_3b_2)]$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + k \begin{vmatrix} a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + k(0) \text{ because row 1st and 2nd are identical}$$

$|B| = |A|$

7. The determinant of a diagonal matrix is equal to the product of its diagonal elements. For example, if

$$|A| = \begin{vmatrix} 2 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 3 \end{vmatrix}$$

$= 2(-15 - 0) - (0 - 0) + 0(0 - 0)$

$= 30$, which is the product of diagonal elements.

i.e., $2(-5)3 = -30$

8. The determinant of the product of two matrices is equal to the product of the determinants of the two matrices, that is $|AB| = |A||B|$. for example, if

$$A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \qquad B = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}$$

$$\text{Then } AB = \begin{vmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{vmatrix}$$

$|AB| = (a_{11}b_{11} + a_{12}b_{21})(a_{21}b_{12} + a_{22}b_{22})$

$- (a_{11}b_{12} + a_{12}b_{22} - a_{11}b_{22})(a_{21}b_{11} + a_{22}b_{21})$

$= a_{11}b_{11}\, a_{21}b_{12} + a_{11}b_{11}\, a_{22}b_{22} + a_{12}b_{21}a_{21}b_{12}$

$+ a_{12}b_{21}\, a_{22}b_{22} - a_{11}b_{12}\, a_{21}b_{11} - a_{11}b_{12}\, a_{22}b_{21}$

$- a_{12}b_{22}\, a_{21}b_{11} - a_{12}b_{22}\, a_{22}b_{21}$

$|AB| = a_{11}b_{11}\, a_{22}b_{22} + a_{12}b_{21}\, a_{21}b_{12} - a_{11}b_{12}\, a_{22}b_{21}$

$- a_{12}b_{22}\, a_{21}b_{11}$ …………………. (A)

and $|A| = a_{11}a_{22} - a_{12}a_{21}$

$|B| = b_{11}b_{22} - b_{12}b_{21}$

$|A|\,|B| = a_{11}b_{11}\, a_{22}\, b_{22} + a_{12}b_{21}\, a_{21}\, b_{12} - a_{11}b_{12}\, a_{22}\, b_{21}$

$- a_{12}b_{22}\, a_{21}\, b_{11}$ ………………… (B)

R.H.S of equations (A) and (B) are equal, so

$|AB| = |A|\,|B|$

9. The determinant in which each element in any row, or column, consists of two terms, then the determinant can be expressed as the sum of two other determinants

$$\begin{vmatrix} a_1+\alpha_1 & b_1 & c_1 \\ a_2+\alpha_2 & b_2 & c_2 \\ a_3+\alpha_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \\ \alpha_3 & b_3 & c_3 \end{vmatrix}$$

Expand by first column.

Proof:

L.H.S $= (a_1+\alpha_1)(b_2c_3-b_3c_2)-(a_2+\alpha_2)(b_1c_3-b_3c_1)+(a_3+\alpha_3)(b_1c_2-b_2c_1)$

$= [(a_1(b_2c_3-b_3c_2)-a_2(b_1c_3-b_3c_1)+a_3(b_1c_2-b_2c_1)]$

$+ [(\alpha_1(b_2c_3-b_3c_2)-\alpha_2(b_1c_2-b_3c_1)+\alpha_3(b_1c_2-b_2c_1)]$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \\ \alpha_3 & b_3 & c_3 \end{vmatrix}$$

= R.H.S

Similarly

$$\begin{vmatrix} \alpha_1+a_1 & b_1+\beta_1 & c_1 \\ \alpha_2+a_2 & b_2+\beta_2 & c_2 \\ \alpha_3+a_3 & b_3+\beta_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & \beta_1 & c_1 \\ a_2 & \beta_2 & c_2 \\ a_3 & \beta_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \\ \alpha_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & \beta_1 & c_1 \\ \alpha_2 & \beta_2 & c_2 \\ \alpha_3 & \beta_3 & c_3 \end{vmatrix}$$

And, $\begin{vmatrix} a_1+\alpha_1 & b_1+\beta_1 & c_1+\gamma_1 \\ a_2+\alpha_2 & b_2+\beta_2 & c_2+\gamma_2 \\ a_3+\alpha_3 & b_3+\beta_3 & c_3+\gamma_3 \end{vmatrix}$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \text{sum of six determinant} + \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$$

Also $\begin{vmatrix} a_1+\alpha_1 & b_1+\beta_1 & c_1+\gamma_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ a_2 & \beta_2 & \gamma_2 \\ a_3 & \beta_3 & \gamma_3 \end{vmatrix}$

Example 5: Verify that $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

Solution:

Multiply row first, second and third by a, b and c respectively, in the L.H.S., then

$$\text{L.H.S} = \frac{1}{abc}\begin{vmatrix} a & a^2 & abc \\ b & b^2 & abc \\ c & c^2 & abc \end{vmatrix}$$

Take abc common from 3rd column

$$= \frac{abc}{abc}\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}$$

Interchange column first and third

$$= -\begin{vmatrix} 1 & a^2 & a \\ 1 & b^2 & b \\ 1 & c^2 & c \end{vmatrix}$$

Again interchange column second and third

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

= R.H.S

Example 6: Show that

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (b-c)\,(c-a)\,(a-b)$$

Solution:

$$\text{L.H.S} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

subtracting row first from second and third row

$$= \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}$$

from row second and third taking (b – a) and (c – a) common.

$$= (b-a)(c-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix}$$

expand from first column

$= (b - a)(c - a)(c + a - b - a)$

$= (b - a)(c - a)(c - b)$

Or L.H.S $= (b - c)(c - a)(a - b)(-1)(-1)$

$= (b - c)(c - a)(a - b) =$ R.H.S

Example 7: Without expansion, show that

$$\begin{vmatrix} 6 & 1 & 3 & 2 \\ -2 & 0 & 1 & 4 \\ 3 & 6 & 1 & 2 \\ -4 & 0 & 2 & 8 \end{vmatrix} = 0$$

Solution:

In the L.H.S Taking 2 common from fourth row, so

$$\text{L.H.S} = 2\begin{vmatrix} 6 & 1 & 3 & 2 \\ -2 & 0 & 1 & 4 \\ 3 & 6 & 1 & 2 \\ -2 & 0 & 1 & 4 \end{vmatrix}$$

Since rows 2nd and 3rd are identical, so

$= 2(0) = 0$

L.H.S = R.H.S

9.8 Solution of Linear Equations by Determinants: (Cramer's Rule)

Consider a system of linear equations in two variables x and y,

$$a_1x + b_1y = c_1 \qquad (1)$$

$$a_2x + b_2y = c_2 \qquad (2)$$

Multiply equation (1) by b_2 and equation (2) by b_1 and subtracting, we get

$$x(a_1b_2 - a_2b_1) = b_2c_1 - b_1c_2$$

$$x = \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1} \qquad (3)$$

Again multiply eq. (1) by a_2 and eq. (2) by a_1 and subtracting, we get

$$y(a_2b_1 - a_1b_2) = a_2c_1 - a_1c_2$$

$$y = \frac{a_2c_1 - a_1c_2}{a_2b_1 - a_1b_2}$$

$$y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1} \qquad (4)$$

Note that x and y from equations (3) and (4) has the same denominator $a_1b_2 - a_2b_1$. So the system of equations (1) and (2) has solution only when $a_1b_2 - a_2b_1 \neq 0$.

The solutions for x and y of the system of equations (1) and (2) can be written directly in terms of determinants without any algebraic operations, as

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \text{ and } y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

This result is called Cramer's Rule.

Here $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = |A|$ is the determinant of the coefficient of x and y in equations (1) and (2)

If $\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} = |A|$

and $\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = |A|$

Then $x = \dfrac{|A_x|}{|A|}$ and $y = \dfrac{|A_y|}{|A|}$

Solution for a system of Linear Equations in Three Variables:

Consider the linear equations:

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Hence the determinant of coefficients is

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \text{ if } |A| \neq 0$$

Then by Cramer's Rule the value of variables is:

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{|A|} = \frac{|A_x|}{|A|}$$

$$y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{|A|} = \frac{|A_y|}{|A|}$$

and $$z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{|A|} = \frac{|A_z|}{|A|}$$

Example 8: Use Cramer's rule to solve the system

$$-4x + 2y - 9z = 2$$
$$3x + 4y + z = 5$$
$$x - 3y + 2z = 8$$

Solution:

Here the determinant of the coefficients is:

$$|A| = \begin{vmatrix} -4 & 2 & -9 \\ 3 & 4 & 1 \\ 1 & -3 & 2 \end{vmatrix}$$

$$= -4(8 + 3) - 2(6 - 1) - 9(-9 - 4)$$

$$= -44 - 10 + 117$$

$$|A| = 63$$

for $|A_x|$, replacing the first column of $|A|$ with the corresponding constants 2, 5 and 8, we have

$$|A_x| = \begin{vmatrix} 2 & 2 & -9 \\ 5 & 4 & 1 \\ 8 & -3 & 2 \end{vmatrix}$$

$$= 2(11) - 2(2) - 9(-47) = 22 - 4 + 423$$

$$\boxed{|A_x| = 441}$$

Similarly,

$$|A_y| = \begin{vmatrix} -4 & 2 & -9 \\ 3 & 5 & 1 \\ 1 & 8 & 2 \end{vmatrix}$$

$$= -4(2) - 2(5) - 9(19)$$

$$= -8 - 10 - 171$$

$$\boxed{|A_y| = -189}$$

and

$$|A_z| = \begin{vmatrix} -4 & 2 & 2 \\ 3 & 4 & 5 \\ 1 & -3 & 8 \end{vmatrix}$$

$$= -4(47) - 2(19) + 2(-13)$$

$$= -188 - 38 - 26$$

$$\boxed{|A_z| = -252}$$

Hence $x = \frac{|A_x|}{|A|} = \frac{441}{63} = 7$

$$y = \frac{|A_y|}{|A|} = \frac{-189}{63} = -3$$

$$z = \frac{|A_z|}{|A|} = \frac{-252}{63} = -4$$

So the solution set of the system is {(7, –3, –4)}

Exercise 9.2

Q.1 Expand the determinants

(i) $\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \\ -2 & 1 & 3 \end{vmatrix}$ (ii) $\begin{vmatrix} a & b & 1 \\ a & b & 1 \\ 1 & 1 & 1 \end{vmatrix}$

(iii) $\begin{vmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{vmatrix}$

Q.2 Without expansion, verify that

(i) $\begin{vmatrix} -2 & 1 & 0 \\ 3 & 4 & 1 \\ -4 & 2 & 0 \end{vmatrix} = 0$ (ii) $\begin{vmatrix} 1 & 2 & 1 \\ 0 & 2 & 3 \\ 2 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & -5 & 0 \end{vmatrix}$

(iii) $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$ (iv) $\begin{vmatrix} bc & ca & ab \\ a^3 & b^3 & c^3 \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \end{vmatrix} = 0$

(v) $\begin{vmatrix} x+1 & x+2 & x+3 \\ x+4 & x+5 & x+6 \\ x+7 & x+8 & x+9 \end{vmatrix} = 0$

(vi) $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} = \begin{vmatrix} e & b & h \\ d & a & g \\ f & c & k \end{vmatrix}$

Q.3 Show that

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1x+d_1 & c_2x+d_2 & c_3x+d_3 \end{vmatrix} = x\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix}$$

Q.4 Show that

(i) $\begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix} = 0$ (ii) $\begin{vmatrix} a & b & c \\ a & a+b & a+b+c \\ a & 2a+b & 3a+2b+c \end{vmatrix} = a^3$

(iii) $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

(iv) $\begin{vmatrix} 1 & 1 & 1 \\ bc & ca & ab \\ b+c & c+a & a+b \end{vmatrix} = (b-c)(c-a)(a-b)$

Q.5 Show that:

(i) $\begin{vmatrix} \ell & a & a \\ a & \ell & a \\ a & a & \ell \end{vmatrix} = (2a+\ell)(\ell-a)^2$

(ii) $\begin{vmatrix} a+\ell & a & a \\ a & a+\ell & a \\ a & a & a+\ell \end{vmatrix} = \ell^2(3a+\ell)$

Q.6 prove that:

(i) $\begin{vmatrix} a & b+c & a+b \\ b & c+a & b+c \\ c & a+b & c+a \end{vmatrix} = a^3 + b^3 + c^3 - 3abc$

(ii) $\begin{vmatrix} a+\lambda & b & c \\ a & b+\lambda & c \\ a & b & c+\lambda \end{vmatrix} = \lambda^2(a+b+c+\lambda)$

(iii) $\begin{vmatrix} \text{Sin }\alpha & \text{Cos }\alpha & 0 \\ -\text{Sin }\beta & \text{Cos }\beta & \text{Sin }\gamma \\ \text{Cos }\beta & \text{Sin }\beta & \text{Cos }\gamma \end{vmatrix} = \text{Sin}(\alpha+\beta+\gamma)$

Q.7 Find values of x if

(i) $\begin{vmatrix} 3 & 1 & x \\ -1 & 3 & 4 \\ x & 1 & 0 \end{vmatrix} = -30$ (ii) $\begin{vmatrix} 1 & 2 & 1 \\ 2 & x & 2 \\ 3 & 6 & x \end{vmatrix} = 0$

Q.8 Use Cramer's rule to solve the following system of equations.

(i) $x - y = 2$
$x + 4y = 5$

(ii) $3x - 4y = -2$
$x + y = 6$

(iii) $x - 2y + z = -1$
$3x + y - 2z = 4$
$y - z = 1$

(iv) $2x + 2y + z = 1$
$x - y + 6z = 21$
$3x + 2y - z = -4$

(v) $x + y + z = 0$
$2x - y - 4z = 15$
$x - 2y - z = 7$

(vi) $x - 2y - 2z = 3$
$2x - 4y + 4z = 1$
$3x - 3y - 3z = 4$

Answers 9.2

Q.1 (i) –41 (ii) 0 (iii) x^3

Q.7 (i) x = -2 , 3 (ii) x = 3 , 4

Q.8 (i) $\left\{\left(\frac{13}{5}, \frac{3}{5}\right)\right\}$ (ii) $\left\{\left(\frac{22}{7}, \frac{20}{7}\right)\right\}$

(iii) {(1, 1, 0)} (iv) {(1, –2, 3)}

(v) {(3, –1, –2)} (vi) $\left\{\left(-\frac{1}{3}, -\frac{25}{24}, -\frac{5}{8}\right)\right\}$