Chapter 9
Matrices and Determinants

9.1 Introduction:

In many economic analysis, variables are assumed to be related by sets of linear equations. Matrix algebra provides a clear and concise notation for the formulation and solution of such problems, many of which would be complicated in conventional algebraic notation. The concept of determinant and is based on that of matrix. Hence we shall first explain a matrix.

9.2 Matrix:

A set of mn numbers (real or complex), arranged in a rectangular formation (array or table) having m rows and n columns and enclosed by a square bracket [ ] is called $m \times n$ matrix (read "m by n matrix") .

An $m \times n$ matrix is expressed as

$$A = \begin{bmatrix} a_{11} & a_{12} & ----a_{1n} \\ a_{21} & a_{22} & ----a_{2n} \\ --- & ---- & ------ \\ --- & ---- & ------ \\ a_{m1} & a_{m2} & ----a_{mn} \end{bmatrix}$$

The letters a_{ij} stand for real numbers. Note that a_{ij} is the element in the ith row and jth column of the matrix .Thus the matrix A is sometimes denoted by simplified form as (a_{ij}) or by $\{a_{ij}\}$ i.e., $A = (a_{ij})$

Matrices are usually denoted by capital letters A, B, C etc and its elements by small letters a, b, c etc.

Order of a Matrix:

The order or dimension of a matrix is the ordered pair having as first component the number of rows and as second component the number of columns in the matrix. If there are 3 rows and 2 columns in a matrix, then its order is written as (3, 2) or (3 x 2) read as three by two. In general if m are rows and n are columns of a matrix, then its order is (m x n).

Examples:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ and } \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{bmatrix}$$

are matrices of orders (2 x 3), (3 x 1) and (4 x 4) respectively.

9.3 Some types of matrices:

1. Row Matrix and Column Matrix:

A matrix consisting of a single row is called a **row matrix or a row vector**, whereas a matrix having single column is called a **column matrix or a column vector.**

2. Null or Zero Matrix:

A matrix in which each element is '0' is called a Null or Zero matrix. Zero matrices are generally denoted by the symbol O. This distinguishes zero matrix from the real number 0.

For example $O = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is a zero matrix of order 2 x 4.

The matrix O_{mxn} has the property that for every matrix A_{mxn},

$A + O = O + A = A$

3. Square matrix:

A matrix A having same numbers of rows and columns is called a square matrix. A matrix A of order m x n can be written as A_{mxn}. If m = n, then the matrix is said to be a square matrix. A square matrix of order n x n, is simply written as A_n.

Thus $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ and $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ are square matrix of order 2 and 3

Main or Principal (leading)Diagonal:

The principal diagonal of a square matrix is the ordered set of elements a_{ij}, where i = j, extending from the upper left-hand corner to the lower right-hand corner of the matrix. Thus, the principal diagonal contains elements a_{11}, a_{22}, a_{33} etc.

For example, the principal diagonal of

$$\begin{bmatrix} 1 & 3 & -1 \\ 5 & 2 & 3 \\ 6 & 4 & 0 \end{bmatrix}$$

consists of elements 1, 2 and 0, in that order.

Particular cases of a square matrix:

(a)Diagonal matrix:

A square matrix in which all elements are zero except those in the main or principal diagonal is called a diagonal matrix. Some elements of the principal diagonal may be zero but not all.

For example $\begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

are diagonal matrices.

In general $A = \begin{bmatrix} a_{11} & a_{12} & --- & a_{1n} \\ a_{21} & a_{22} & --- & a_{2n} \\ --- & --- & --- & --- \\ --- & --- & --- & --- \\ a_{n1} & a_{n2} & --- & a_{nn} \end{bmatrix} = (a_{ij})_{nxn}$

is a diagonal matrix if and only if

$a_{ij} = 0$ for $i \neq j$

$a_{ij} \neq 0$ for at least one $i = j$

(b) Scalar Matrix:

A diagonal matrix in which all the diagonal elements are same, is called a scalar matrix i.e.

Thus

$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ and $\begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$ are scalar matrices

(c) Identity Matrix or Unit matrix:

A scalar matrix in which each diagonal element is 1(unity) is called a unit matrix. An identity matrix of order n is denoted by I_n.

Thus $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

are the identity matrices of order 2 and 3 .

In general, $A = \begin{bmatrix} a_{11} & a_{12} & ----a_{1n} \\ a_{21} & a_{22} & ----a_{2n} \\ --- & ---- & ------- \\ --- & ---- & ------- \\ a_{m1} & a_{m2} & ----a_{mn} \end{bmatrix} = [a_{ij}]_{mxn}$

is an identity matrix if and only if

$a_{ij} = 0$ for $i \neq j$ and $a_{ij} = 1$ for $i = j$

Note: If a matrix A and identity matrix I are comformable for multiplication, then I has the property that

AI = IA = A i.e., I is the identity matrix for multiplication.

4. Equal Matrices:

Two matrices A and B are said to be equal if and only if they have the same order and each element of matrix A is equal to the corresponding element of matrix B i.e for each i, j, $a_{ij} = b_{ij}$

Thus $A = \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} \frac{4}{2} & 2-1 \\ \sqrt{9} & 0 \end{bmatrix}$

then A = B because the order of matrices A and B is same and $a_{ij} = b_{ij}$ for every i , j.

Example 1: Find the values of x , y , z and a which satisfy the matrix equation

$$\begin{bmatrix} x+3 & 2y+x \\ z-1 & 4a-6 \end{bmatrix} = \begin{bmatrix} 0 & -7 \\ 3 & 2a \end{bmatrix}$$

Solution : By the definition of equality of matrices, we have

x + 3 = 0 ……………………………..(1)

2y + x = -7 ……………………………(2)

z – 1 = 3 ……………………………(3)

4a – 6 = 2a ……………………………(4)

From (1) x = -3

Put the value of x in (2) , we get $y = -2$
From (3) $z = 4$
From (4) $a = 3$

5. The Negative of a Matrix:

The negative of the matrix A_{mxn}, denoted by $-A_{mxn}$, is the matrix formed by replacing each element in the matrix A_{mxn} with its additive inverse. For example,

If $$A_{3x2} = \begin{bmatrix} 3 & -1 \\ 2 & -2 \\ -4 & 5 \end{bmatrix}$$

Then $$-A_{3x2} = \begin{bmatrix} -3 & 1 \\ -2 & 2 \\ 4 & -5 \end{bmatrix}$$

for every matrix A_{mxn}, the matrix $-A_{mxn}$ has the property that

$A + (-A) = (-A) + A = 0$

i.e., $(-A)$ is the additive inverse of A.

The sum $B_{m-n} + (-A_{mxn})$ is called the difference of B_{mxn} and A_{mxn} and is denoted by $B_{mxn} - A_{mxn}$.

9.4 Operations on matrices:

(a) Multiplication of a Matrix by a Scalar:

If A is a matrix and k is a scalar (constant), then kA is a matrix whose elements are the elements of A , each multiplied by k

For example, if $A = \begin{bmatrix} 4 & -3 \\ 8 & -2 \\ -1 & 0 \end{bmatrix}$ then for a scalar k,

$$kA = \begin{bmatrix} 4k & -3k \\ 8k & -2k \\ -k & 0 \end{bmatrix}$$

Also, $$3\begin{bmatrix}5 & -8 & 4\\0 & 3 & -5\\3 & -1 & 4\end{bmatrix}=\begin{bmatrix}15 & -24 & 12\\0 & 9 & -15\\9 & -3 & 12\end{bmatrix}$$

(b) Addition and subtraction of Matrices:

If A and B are two matrices of same order m×n then their sum A + B is defined as C, m×n matrix such that each element of C is the sum of the corresponding elements of A and B .

for example

If $A=\begin{bmatrix}3 & 1 & 2\\2 & 1 & 4\end{bmatrix}$ and $B=\begin{bmatrix}1 & 0 & 2\\-1 & 3 & 0\end{bmatrix}$

Then $C = A + B = \begin{bmatrix}3+1 & 1+0 & 2+2\\2-1 & 1+3 & 4+0\end{bmatrix}=\begin{bmatrix}4 & 1 & 4\\1 & 4 & 4\end{bmatrix}$

Similarly, the difference A – B of the two matrices A and B is a matrix each element of which is obtained by subtracting the elements of B from the corresponding elements of A

Thus if $A=\begin{bmatrix}6 & 2\\7 & -5\end{bmatrix}$, $B=\begin{bmatrix}8 & 1\\3 & 4\end{bmatrix}$

then $$A - B = \begin{bmatrix}6 & 2\\7 & -5\end{bmatrix}-\begin{bmatrix}8 & 1\\3 & 4\end{bmatrix}=\begin{bmatrix}6-8 & 2-1\\7-3 & -5-4\end{bmatrix}$$
$$=\begin{bmatrix}-2 & 1\\4 & -9\end{bmatrix}$$

If A, B and C are the matrices of the same order mxn

then A + B = B + A

and (A + B) + C = A + (B + C) i.e., the addition of matrices is commutative and Associative respectively.

Note: The sum or difference of two matrices of different order is not defined.

(c) Product of Matrices:

Two matrices A and B are said to be conformable for the product AB if the number of columns of A is equal to the number of rows of B. Then the product matrix AB has the same number of rows as A and the same number of columns as B.

Thus the product of the matrices A_{mxp} and B_{pxn} is the matrix $(AB)_{mxn}$. The elements of AB are determined as follows:

The element C_{ij} in the ith row and jth column of $(AB)_{mxn}$ is found by $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \dots\dots\dots + a_{in}b_{nj}$

for example, consider the matrices

$$A_{2x2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad B_{2x2} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

Since the number of columns of A is equal to the number of rows of B ,the product AB is defined and is given as

$$AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

Thus c_{11} is obtained by multiplying the elements of the first row of A i.e., a_{11} , a_{12} by the corresponding elements of the first column of B i.e., b_{11} , b_{21} and adding the product.

Similarly , c_{12} is obtained by multiplying the elements of the first row of A i.e., a_{11} , a_{12} by the corresponding elements of the second column of B i.e., b_{12} , b_{22} and adding the product. Similarly for c_{21} , c_{22} .

Note :

1 . Multiplication of matrices is not commutative i.e., AB $\neq$ BA in general.
2 . For matrices A and B if AB = BA then A and B commute to each other
3 . A matrix A can be multiplied by itself if and only if it is a square matrix.The product A.A in such cases is written as A^2.
Similarly we may define higher powers of a square matrix i.e., $A . A^2 = A^3$, $A^2. A^2 = A^4$
4. In the product AB, A is said to be pre multiple of B and B is said to be post multiple of A.

Example 1: **If** $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$ **and** $B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ **Find AB and BA.**

Solution:

$$AB = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2+2 & 1+2 \\ -2+3 & -1+3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2-1 & 4+3 \\ 1-1 & 2+3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 7 \\ 0 & 5 \end{bmatrix}$$

This example shows very clearly that multiplication of matrices in general, is not commutative i.e., $AB \neq BA$.

Example 2: If

Example 2: **If** $A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$ **and** $B = \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}$**, find AB**

Solution:

Since A is a (2 x 3) matrix and B is a (3 x 2) matrix, they are conformable for multiplication. We have

$$AB = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 3+2+6 & -3+1+2 \\ 1+0+3 & -1+0+1 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 0 \\ 4 & 0 \end{bmatrix}$$

Remark:

If A, B and C are the matrices of order (m x p), (p x q) and (q x n) respectively, then

i. $(AB)C = A(BC)$ i.e., Associative law holds.

ii. $C(A+B) = CA + CB$ and $(A + B)C = AC + BC$ }i.e distributive laws holds.

Note: that if a matrix A and identity matrix I are conformable for multiplication, then I has the property that

$AI = IA = A$ i.e, I is the identity matrix for multiplication.

Exercise 9.1

Q.No. 1 Write the following matrices in tabular form:

i. $A = [a_{ij}]$, where $i = 1, 2, 3$ and $j = 1, 2, 3, 4$

ii. $B = [b_{ij}]$, where $i = 1$ and $j = 1, 2, 3, 4$

iii. $C = [c_{jk}]$, where $j = 1, 2, 3$ and $k = 1$

Q.No.2 Write each sum as a single matrix:

i. $\begin{bmatrix} 2 & 1 & 4 \\ 3 & -1 & 0 \end{bmatrix} + \begin{bmatrix} 6 & 3 & 0 \\ -2 & 1 & 0 \end{bmatrix}$

ii. $\begin{bmatrix} 1 & 3 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 0 & -2 & 1 & 3 \end{bmatrix}$

iii. $\begin{bmatrix} 4 \\ 3 \\ -1 \end{bmatrix} + \begin{bmatrix} 6 \\ 0 \\ -2 \end{bmatrix}$

iv. $\begin{bmatrix} 2 & 3 & 4 \\ -1 & 6 & 2 \\ 1 & 0 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

v. $2\begin{bmatrix} 6 & 1 \\ 0 & -3 \\ -1 & 2 \end{bmatrix} - 3\begin{bmatrix} 4 & 2 \\ 0 & 1 \\ -5 & -1 \end{bmatrix}$

Q.3 Show that $\begin{bmatrix} b_{11} - a_{11} & b_{12} - a_{12} \\ b_{21} - a_{21} & b_{22} - a_{22} \end{bmatrix}$ is a solution of the matrix equation $X + A = B$, where $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$.

Q.4 Solve each of the following matrix equations:

i. $X + \begin{bmatrix} 3 & -1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ -3 & 1 \end{bmatrix}$

ii. $X + \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ 1 & 5 \end{bmatrix} + \begin{bmatrix} -4 & -8 \\ -2 & 0 \end{bmatrix}$

iii. $3X + \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \\ 4 & -1 & 5 \end{bmatrix} = \begin{bmatrix} -2 & 3 & 1 \\ -1 & -2 & 0 \\ 0 & 1 & 5 \end{bmatrix}$

iv. $X + 2I = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}$

Q.5 Write each product as a single matrix:

i. $\begin{bmatrix} 3 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 1 & 0 \end{bmatrix}$

ii. $[3 \quad -2 \quad 2]\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$

iii. $\begin{bmatrix} 2 & -2 & -1 \\ 1 & 1 & -2 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & -2 & 5 \\ -1 & -1 & 3 \\ -1 & -2 & 4 \end{bmatrix}$

iv. $\begin{bmatrix} -1 & -2 & 5 \\ -1 & -1 & 3 \\ -1 & -2 & 4 \end{bmatrix}\begin{bmatrix} 2 & -2 & -1 \\ 1 & 1 & -2 \\ 1 & 0 & -1 \end{bmatrix}$

Q.6 If $A = \begin{bmatrix} 1 & 4 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 2 \\ 4 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, find $A^2 + BC$.

Q.7 Show that if $A = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$, then

(a) $(A + B)(A + B) \neq A^2 + 2AB + B^2$

(b) $(A + B)(A - B) \neq A^2 - B^2$

Q.8 Show that:

(i) $\begin{bmatrix} -1 & 2 & 3 \\ 2 & 1 & 0 \\ 3 & 5 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -a + 2b + 3c \\ 2a + b \\ 3a + 5b - c \end{bmatrix}$

(ii) $\begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ +\sin\theta & 0 & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & 0 & +\sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Q.9 If $A = \begin{bmatrix} 2 & -2\sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 2\sqrt{2} \\ -\sqrt{2} & 2 \end{bmatrix}$

Show that A and B commute.

Answers 9.1

Q.1(i) $\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$

(ii) $\begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \end{bmatrix}$

(iii) $\begin{bmatrix} c_{11} \\ c_{21} \\ c_{31} \end{bmatrix}$

Q.2 (i) $\begin{bmatrix} 8 & 4 & 4 \\ 1 & 0 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 1 & 6 & 9 \end{bmatrix}$

(iii) $\begin{bmatrix} 10 \\ 3 \\ -3 \end{bmatrix}$ (iv) $\begin{bmatrix} 2 & 3 & 4 \\ -1 & 6 & 2 \\ 1 & 0 & 3 \end{bmatrix}$

(v) $\begin{bmatrix} 0 & 4 & 0 \\ -9 & 13 & 7 \end{bmatrix}$

Q.4 (i) $\begin{bmatrix} 2 & 2 \\ -5 & -1 \end{bmatrix}$ (ii) $\begin{bmatrix} -1 & -2 \\ -1 & 3 \end{bmatrix}$

(iii) $\begin{bmatrix} -1 & 1 & -\frac{1}{3} \\ -1 & -1 & -1 \\ -\frac{4}{3} & \frac{2}{3} & 0 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$

Q.5 (i) $\begin{bmatrix} 2 & -1 \\ 2 & -2 \end{bmatrix}$ (ii) $\begin{bmatrix} -1 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$