Chapter 8
Vectors and Scalars

8.1 Introduction:

In this chapter we shall use the ideas of the plane to develop a new mathematical concept, vector. If you have studied physics, you have encountered this concept in that part of physics concerned with forces and equilibrium.

Physicists were responsible for first conceiving the idea of a vector, but the mathematical concept of vectors has become important in its own right and has extremely wide application, not only in the sciences but in mathematics as well.

8.2 Scalars and Vectors:

A quantity which is completely specified by a certain number associated with a suitable unit without any mention of direction in space is known as scalar. Examples of scalar are time, mass, length, volume, density, temperature, energy, distance, speed etc. The number describing the quantity of a particular scalar is known as its magnitude. The scalars are added subtracted, multiplied and divided by the usual arithmetical laws.

A quantity which is completely described only when both their magnitude and direction are specified is known as vector. Examples of vector are force, velocity, acceleration, displacement, torque, momentum, gravitational force, electric and magnetic intensities etc. A vector is represented by a Roman letter in bold face and its magnitude, by the same letter in italics. Thus **V** means vector and *V* is magnitude.

8.3 Vector Representations:

A vector quantity is represented by a straight line segment, say $\overrightarrow{PQ}$. The arrow head indicate the direction from P to Q. The length of the Vector represents its magnitude. Sometimes the vectors are represented by single letter such as V or $\vec{V}$. The magnitude of a vector is denoted by |V| or by just **V**, where $|\vec{V}|$ means modulus of $\vec{V}$ which is a positive value

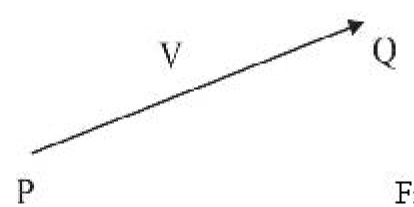


Fig. 1

8.4 Types of Vectors:

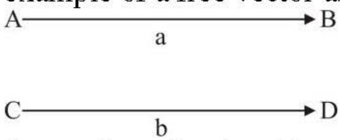
1. Unit Vector:

A vector whose magnitude is unity i.e., 1 and direction along the given vector is called a unit Vector. If $\overline{a}$ is a vector then a unit vector in the direction of $\overline{a}$, denoted by $\hat{a}$ (read as a cap), is given as,

$$\hat{a} = \frac{\overline{a}}{|a|} \quad \text{or} \quad \overline{a} = |a|\,\hat{a}$$

2. Free Vector:

A vector whose position is not fixed in space. Thus, the line of action of a free vector can be shifted parallel to itself. Displacement is an example of a free vector as shown in figure 1:



3. Localized or Bounded Vectors:

A vector which cannot be shifted parallel to itself, i.e., whose line of action is fixed is called a localized or bounded vector. Force and momentum are examples of localized vectors.

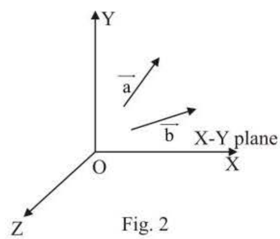


Fig. 2

4. Coplanar Vectors:

The vectors which lies in the same plane are called coplanar vectors, as shown in Fig. 2.

5. Concurrent Vectors:

The vectors which pass through the common point are called concurrent vectors. In the figure no.3 vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are called concurrent as they pass through the same point.

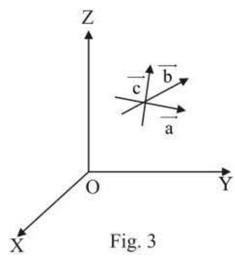


Fig. 3

6. Negative of a Vector:

The vector which has the same magnitude as the vector $\overline{a}$ but opposite in direction to $\overline{a}$ is called the negative to $\overline{a}$. It is represented by $-\overline{a}$. Thus of $\overline{AB} = -\overline{a}$ then $\overline{BA} = -\overline{a}$

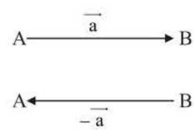


Fig. 4

7. Null or Zero Vector:

It is a vector whose magnitude is zero. We denote the null vector by O. The direction of a zero vector is arbitrary.

The vectors other than zero vectors are proper vectors or non-zero vectors.

8. Equal Vectors:

Two vectors $\vec{a}$ and $\vec{b}$ are said to be equal if they have the same magnitude and direction. If $\vec{a}$ and $\vec{b}$ are equal vectors then $\vec{a} = \vec{b}$

9. Parallel and Collinear Vectors:

The vectors $\vec{a}$ and $\vec{b}$ are parallel if for any real number n,

$\vec{a} = n\,\vec{b}$. If

(i) $n > 0$ then the vectors $\vec{a}$ and $\vec{b}$ have the same direction.

(ii) $n < 0$ then $\vec{a}$ and $\vec{b}$ have opposite directions.

Now, we can also define collinear vectors which lie along the same straight line or having their directions parallel to one another.

10. Like and Unlike Vectors:

The vectors having same direction are called like vectors and those having opposite directions are called unlike vectors.

11. Position Vectors (PV):

If vector $\overrightarrow{OA}$ is used to specify the position of a point A relative to another point O. This $\overrightarrow{OA}$ is called the position vector of A referred to O as origin. In the figure 4 $\vec{a} = \overrightarrow{OA}$ and $\overrightarrow{OB} = \vec{b}$ are the position vector (P.V) of A and B respectively. The vector $\overrightarrow{AB}$ is determined as follows:

By the head and tail rules,

$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$

Or $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \vec{b} - \vec{a}$

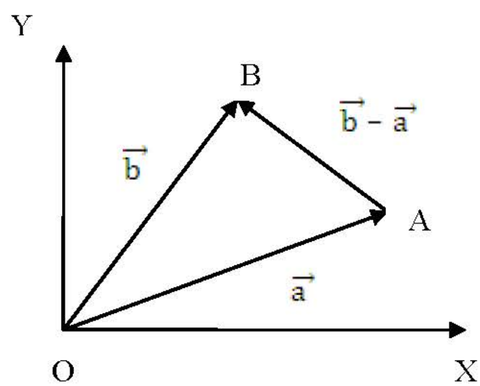


Fig. 5

8.5 Addition and Subtraction of Vectors:

1. Addition of Vectors:

Suppose $\vec{a}$ and $\vec{b}$ are any two vectors. Choose point A so that $\vec{a} = \overrightarrow{OA}$ and choose point C so that $\vec{b} = \overrightarrow{AC}$. The sum, $\vec{a} + \vec{b}$ of $\vec{a}$ and $\vec{b}$ is the vector is the vector $\overrightarrow{OC}$. Thus the sum of two vectors $\vec{a}$ and $\vec{b}$ is performed by the Triangle Law of addition.

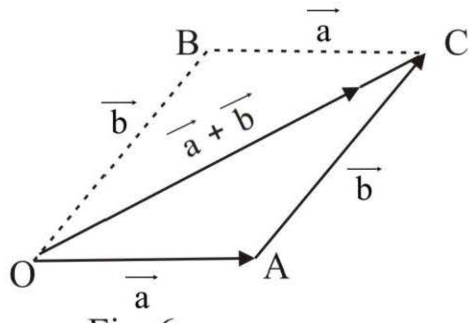


Fig. 6

2. Subtraction of Vectors:

If a vector $\vec{b}$ is to be subtracted from a vector $\vec{a}$, the difference vector $\vec{a} - \vec{b}$ can be obtained by adding vectors $\vec{a}$ and $-\vec{b}$.

The vector $-\vec{b}$ is a vector which is equal and parallel to that of vector but its arrow-head points in opposite direction. Now the vectors $\vec{a}$ and $-\vec{b}$ can be added by the head-to-tail rule. Thus the line $\overrightarrow{AC}$ represents, in magnitude and direction, the vector $\vec{a} - \vec{b}$.

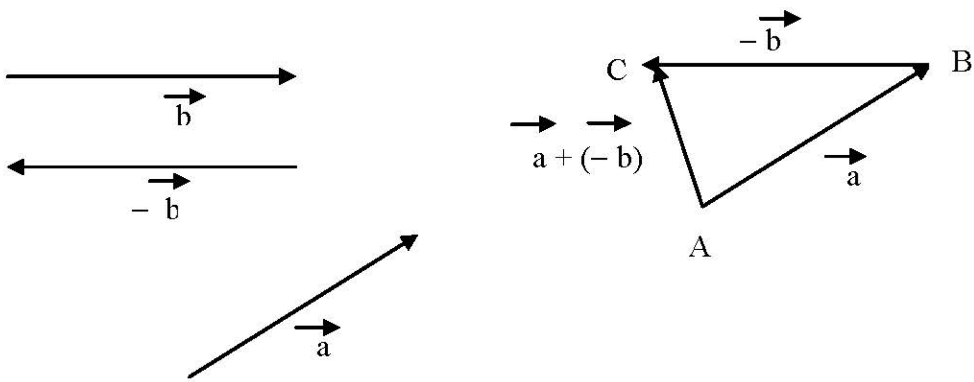


Fig . 7

Properties of Vector Addition:

i. Vector addition is commutative

i.e., $\vec{a} + \vec{b} = \vec{a} + \vec{b}$ where $\vec{a}$ and $\vec{b}$ are any two vectors.

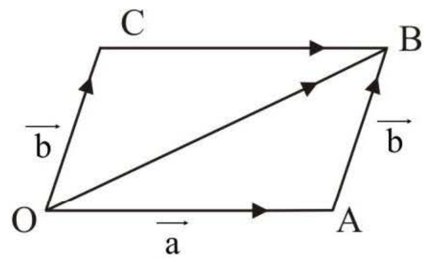


Fig. 8

(ii) Vectors Addition is Associative:

i.e. $$\left(\overrightarrow{a}+\overrightarrow{b}\right)+\overrightarrow{c}=\overrightarrow{a}+\left(\overrightarrow{b}+\overrightarrow{c}\right)$$

where $\overrightarrow{a}$, $\overrightarrow{b}$ and $\overrightarrow{c}$ are any three vectors.

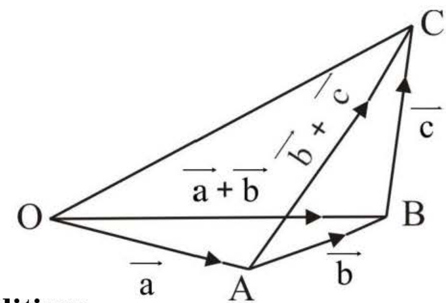


Fig.9

(iii) $\overline{O}$ is the identity in vectors addition:

For every vector $\overrightarrow{a}$

$\overrightarrow{a}+\overline{O}=\overrightarrow{a}$

Where $\overline{O}$ is the zero vector.

Remarks: Non-parallel vectors are not added or subtracted by the ordinary algebraic Laws because their resultant depends upon their directions as well.

8.6 Multiplication of a Vector by a Scalar:

If $\overrightarrow{a}$ is any vectors and K is a scalar, then $K\overrightarrow{a}=\overrightarrow{a}K$ is a vector with magnitude $|K|.\,|a|$ i.e., $|K|$ times the magnitude of $\overrightarrow{a}$ and whose direction is that of vector $\overrightarrow{a}$ or opposite to vector $\overrightarrow{a}$ according as K is positive or negative resp. In particular $\overrightarrow{a}$ and $-\overrightarrow{a}$ are opposite vectors.

Properties of Multiplication of Vectors by Scalars:

1. The scalar multiplication of a vectors satisfies
$$m(n\overrightarrow{a})=(mn)\overrightarrow{a}=n(m\overrightarrow{a})$$
2. The scalar multiplication of a vector satisfies the distributive laws
i.e., $$(m+n)\overrightarrow{a}=m\overrightarrow{a}+n\overrightarrow{a}$$

and $m(\overline{a}+\overline{b}) = m\overline{a} + m\overline{b}$

Where m and n are scalars and $\dot{a}$ and $\dot{b}$ are vectors.

8.7 The Unit Vectors i, j, k (orthogonal system of unit Vectors):

Let us consider three mutually perpendicular straight lines OX, OY and OZ. These three mutually perpendicular lines determine uniquely the position of a point. Hence these lines may be taken as the co-ordinates axes with O as the origin.

We shall use i, j and k to denote the Unit Vectors along OX, OY and OX respectively.

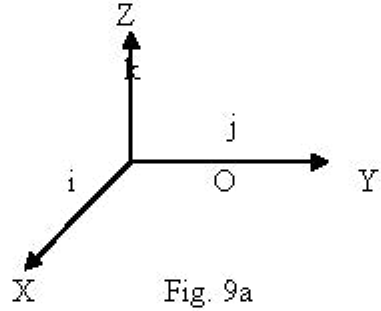


Fig. 9a

8.8 Representation of a Vector in the Form of Unit Vectors i, j and k.

Let us consider a vector $\overline{\mathbf{r}} = \overline{\mathbf{OP}}$ as shown in fig. 11. Then x i, y j and z k are vectors directed along the axes,

$$\overrightarrow{OQ} = \overrightarrow{OA} + \overrightarrow{AQ} = \overrightarrow{OA} + \overrightarrow{OB} \qquad \text{because}$$

and $\dot{OQ} = xi + yi$

Because $\overline{QP} = zk$

$\overline{OP} = \overline{OQ} + \overline{QP}$

and $r = \dot{OP} = xi + yj + zk$

Here the real numbers x, y and z are the components of Vector $\overline{r}$ or the co-ordinates of point P in the direction of OX, OY and OZ respectively. The vectors xi, yj and zk are called the resolved parts of the vector $\dot{r}$ in the direction of the Unit vectors i, j and k respectively.

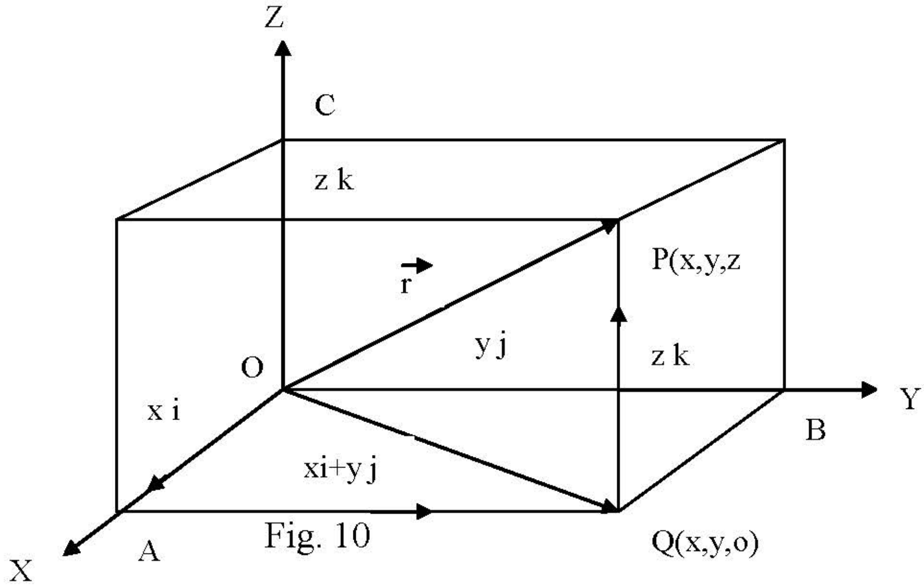


Fig. 10

8.9 Components of a Vector when the Tail is not at the Origin:

Consider a vector $\vec{r} = \overrightarrow{PQ}$ whose tail is at the point $P(x_1, y_1, z_1)$ and the head at the point $Q\ (x_2, y_2, z_2)$. Draw perpendiculars PP′ and QQ′ on x-aixs.

$P'Q' = x_2 - x_1$ = x-component of $\vec{r}$

Now draw perpendiculars PP^o and QQ^o on y-axis.

Then $P^oQ^o = y_2 - y_1$ = y-component of $\vec{r}$

Similarly $z_2 - z_1$ = z-component of $\vec{r}$

Hence the vector $\vec{r}$ can be written as,

$$\vec{r} = \overrightarrow{PQ} = (x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k$$

$$\text{Or, } \vec{r} = \overrightarrow{PQ} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

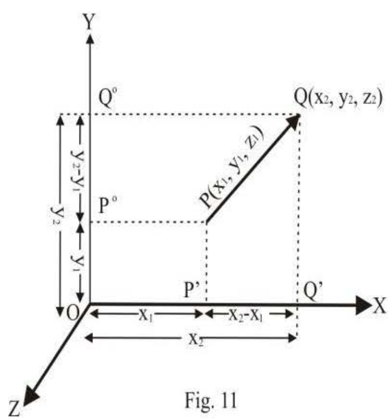


Fig. 11

8.10 Magnitude or Modulus of a Vector:

Suppose x, y and z are the magnitude of the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ as shown in fig. 10.

In the right triangle OAQ, by Pythagorean Theorem

$$OQ^2 = x^2 + y^2$$

Also in the right triangle OQP, we have

$$OP^2 = OQ^2 + QP^2$$
$$OP^2 = x^2 + y^2 + z^2$$

Or $\quad |\overrightarrow{r}| = |OP| = \sqrt{x^2 + y^2 + z^2}$

Thus if $\quad \overrightarrow{r} = \overline{PQ} = xi + yj + zk$

Then, its magnitude is

$$|\overrightarrow{r}| = \sqrt{x^2 + y^2 + z^2}$$

If $\quad \overline{r} = (x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k$

Then $\quad |\overrightarrow{r}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Example 1:

If $P_1 = P(7, 4, -1)$ and $P_2 = P(3, -5, 4)$, what are the components of P_1P_2? Express $\overrightarrow{P_1 P_2}$ in terms of i, j and k.

Solution:

x-component of $P_1 P_2 = x_2 - x_1 = 3 - 7 = -4$

y-component of $P_1 P_2 = y_2 - y_1 = -5 - 4 = -9$

and z-component of $P_1 P_2 = z_2 - z_1 = 4 - (-1) = 5$

also $\quad \overrightarrow{P_1 P_2} = (x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k$

$$\overrightarrow{P_1 P_2} = -4i - 9j + 5k$$

Example 2: Find the magnitude of the vector

$$|\overline{u}| = \frac{3}{5}i - \frac{2}{5}j + \frac{2\sqrt{3}}{5}k$$

Solution:

$$|\overline{u}| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(-\frac{2}{5}\right)^2 + \left(\frac{2\sqrt{3}}{5}\right)^2}$$

$$= \sqrt{\frac{9}{25} + \frac{4}{25} + \frac{12}{25}} = \sqrt{\frac{25}{25}}$$

$$|\overline{u}| = 1$$

Note: Two vectors are equal if and only if the corresponding components of these vectors are equal relative to the same co-ordinate system.

Example 3:

Find real numbers x, y and z such that

$xi + 2yj - zk + 3i - j = 4i + 3k$

Solution:

Since $(x + 3)i + (2y - 1)j + (-z)k = 4i + 3k$

Comparing both sides,we get

$x + 3 = 4,\ 2y - 1 = 0, \quad -z = 3$

$x = 1, \quad y = \frac{1}{2} \quad , \quad z = -3$

Note 2: If

$$\overrightarrow{r_1} = x_1 i + y_1 j + z_1 k$$
$$\overrightarrow{r_2} = x_2 i + y_2 j + z_2 k$$

Then the sum vector =

$$\overrightarrow{r_1} + \overrightarrow{r_2} = (x_1 + x_2)i + (y_1 + y_2)j + (z_1 + z_2)k$$

Or

$$\overrightarrow{r_1} + \overrightarrow{r_2} = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$$

Example 4:

$\overrightarrow{a} = 3i - 2j + 5k$ and $\overrightarrow{b} = -2i - j + k$.

Find $2\overrightarrow{a} - 3\overrightarrow{b}$ and also its unit vector.

Solution:

$$2\overrightarrow{a} - 3\overrightarrow{b} = 2(3i - 2j + 5k) - 3(-2i - j + k)$$
$$= 6i - 4j + 10k + 6i + 3j - 3k$$
$$= 12i - j + 7k$$

If we denote $2\overrightarrow{a} - 3\overrightarrow{b} = \overrightarrow{c}$, then $\overrightarrow{c} = 12i - j + 7k$

and $|\vec{c}| = \sqrt{12^2 + (-1)^2 + 7^2} = \sqrt{144 + 1 + 49} = \sqrt{194}$

Therefore, $\hat{c} = \frac{\overrightarrow{c}}{|c|} = \frac{12i - j + 7k}{\sqrt{194}}$

$$\hat{c} = \frac{12}{\sqrt{194}} i - \frac{1}{\sqrt{194}} j + \frac{1}{\sqrt{194}} k$$

Note 3: Two vectors $\vec{r_1} = x_1 i + y_1 j + z_1 k$ and $\vec{r_2} = x_2 i + y_2 j + z_2 k$ are parallel if and only if $\frac{x_1}{x_2} = \frac{y_1}{y_2} = \frac{z_1}{z_2}$.

8.11 Direction Cosines:

Let us consider that the vector $\vec{r} = \overline{OP}$ which makes angles α, β and γ with the coordinate axes OX, OY and OZ respectively. Then Cos α, Cos β and Cos γ are called the direction cosines of the vector $\overline{OP}$. They are usually denoted by l, m and n respectively.

If $\overline{OP} = \vec{r} = xi + yj + zk$, then x, y and z are defined as the direction ratios of the vector $\vec{r}$ and $\vec{r} = \sqrt{x^2 + y^2 + z^2}$. Since the angles A, B and C are right angles (by the fig. 11), so in the right triangles.

OAP, OBP and OCP the direction cosines of $\vec{r}$ can be written as,

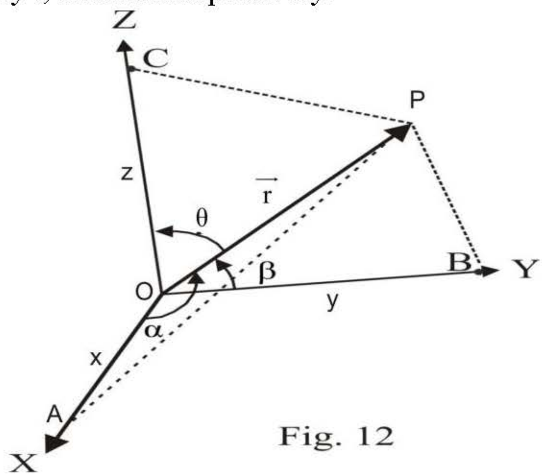


Fig. 12

$$l = \cos\alpha = \frac{x}{|r|} = \frac{x}{\sqrt{x^2 + y^2 + z^2}}$$

$$m = \cos\beta = \frac{y}{|r|} = \frac{y}{\sqrt{x^2 + y^2 + z^2}}$$

and $$n = \cos\gamma = \frac{z}{|r|} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

Note 1: Since the unit vector $\hat{r} = \frac{\vec{r}}{|r|} = \frac{xi + yj + zk}{|r|}$

$$\hat{r} = \frac{x}{|r|}i + \frac{y}{|r|}j + \frac{z}{|r|}k$$

$$\hat{r} = \text{Cos } \alpha \text{ i} + \text{Cos } \beta \text{ j} + \text{Cos } \gamma \text{ k}$$

Or $$\hat{r} = li + mj + nk$$

Therefore the co-efficient of i, j and k in the unit vector are the direction cosines of a vector.

Note 2: $$l^2 + m^2 + n^2 = \frac{x^2}{|r|^2} + \frac{y^2}{|r|^2} + \frac{z^2}{|r|^2}$$

$$= \frac{x^2+y^2+z^2}{|r|^2} = \frac{x^2+y^2+z^2}{x^2+y^2+z^2} = 1$$

Example 5:

Find the magnitude and direction cosines of the vectors
3i + 7j – 4k, i – 5j – 8k and 6i – 2j + 12k.

Solution:

Let $\overrightarrow{a} = 3i + 7j - 4k$

$\overrightarrow{b} = i - 5j - 8k$

$\overrightarrow{c} = 6i - 2j + 12k$

Now $\hat{a} = \frac{\overrightarrow{a}}{|a|} = \frac{3i + 7j - 4k}{\sqrt{74}}$

$$= \frac{3}{\sqrt{74}}i + \frac{7}{\sqrt{74}}j - \frac{4}{\sqrt{74}}k$$

So the direction cosines of $\overrightarrow{a}$ are: $\frac{3}{\sqrt{74}}, \frac{7}{\sqrt{74}}, -\frac{4}{\sqrt{74}}$

Similarly the direction cosines of $\overrightarrow{b}$ are: $\frac{1}{\sqrt{90}}, -\frac{5}{\sqrt{90}}, -\frac{8}{\sqrt{90}}$

and the direction cosines of $\overrightarrow{c}$ are: $\frac{6}{\sqrt{184}}, -\frac{2}{\sqrt{184}}, \frac{12}{\sqrt{184}}$

Exercise8.1

Q.1 If $\overrightarrow{a} = 3i - j - 4k$, $b = -2i + 4j - 3k$ and $\overrightarrow{c} = i + 2j - k$.
Find unit vector parallel to $3\overrightarrow{a} - 2\overrightarrow{b} + 4\overrightarrow{c}$.

Q.2 Find the vector whose magnitude is 5 and which is in the direction of the vector 4i – 3j + k.

Q.3 For what value of m, the vector 4i + 2j – 3k and mi – j + $\sqrt{3}$k have same magnitude?

Q.4 Given the points A = (1, 2, –1), B = (–3, 1, 2) and C = (0, –4, 3)
(i) find $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{AC}$ (ii) Show that $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

Q.5 Find the lengths of the sides of a triangle, whose vertices are A = (2, 4, –1), B = (4, 5, 1), C = (3, 6, –3) and show that the triangle is right angled.

Q.6 If vectors 3i + j – k and λ i – 4j + 4k are parallel, find the value of λ .

Q.7 Show that the vectors 4i – 6j + 9k and –6i + 9j $-\frac{27}{2}$ k are parallel.

Q.8 Find real numbers x, y and z such that
(a) 7xi + (y – 3)j + 6k = 10i + 8j – 3zk
(b) (x + 4)i + (y – 5)j + (z – 1)k = 0

Q.9 Given the vectors $\vec{a} = 3i - 2j + 4k$ and $\vec{b} = 2i + j + 3k$ find the magnitude and direction cosines of
(i) $\vec{a} - \vec{b}$ (ii) $3\vec{a} - 2\vec{b}$

Q.10 If the position vector of $\vec{A}$ and $\vec{B}$ 5i – 2j + 4k and i + 3j + 7k respectively, find the magnitude and direction cosines of $\overrightarrow{AB}$.

Answers 8.1

Q.1 $\frac{1}{\sqrt{398}}(17i-3j-10k)$ Q.2 $\frac{5}{\sqrt{26}}(4i-3j+k)$

Q.3 ±5 Q.4 $(-4,\ -1,\ 3),(3,\ -5,\ 1),(\ -1,\ -6,4)$

Q.5 AB = AC = 3, BC = $3\sqrt{2}$ Q.6 $\lambda = -12$

Q.8 (a) $x=\frac{10}{7}$ y = 11 z = - 2 (b) x = -4 , y = 5 , z = 1

Q.9 (a) $\sqrt{11}; \frac{1}{\sqrt{11}}\ \frac{-3}{\sqrt{11}}\ \frac{1}{\sqrt{11}}$ (b) $5\sqrt{5}; \frac{1}{\sqrt{5}}\ \frac{-8}{5\sqrt{5}}\ \frac{6}{5\sqrt{5}}$

Q.10 $\sqrt{50}; \frac{-4}{\sqrt{50}}, \frac{5}{\sqrt{50}}, \frac{3}{\sqrt{50}}$

8.12 Product of Vectors:

1. Scalar Product of two Vectors:

If $\overline{a}$ and $\overline{b}$ are non-zero vectors, and θ is the angle between them, then the scalar product of $\overline{a}$ and $\overline{b}$ is denoted by $\overline{a} \cdot \overline{b}$ and read as $\overline{a}$ dot $\overline{b}$. It is defined by the relation

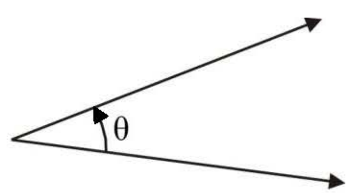


Fig.14

$$\overline{a} \cdot \overline{b} = |a|\ |b| \operatorname{Cos} \theta \qquad \ldots\ldots\ldots\ldots.. \quad (1)$$

If either $\overline{a}$ or $\overline{b}$ is the zero vector, then $\overline{a} \cdot \overline{b} = 0$