6.8 Conversion of sum or difference to products:

We know that

$$\text{Sin}(\alpha+\beta)=\text{Sin }\alpha\text{ Cos }\beta+\text{ Cos }\alpha\text{ Sin }\beta \;\ldots\ldots\; (1)$$

$$\text{Sin}(\alpha-\beta)=\text{Sin }\alpha\text{ Cos }\beta-\text{Cos }\alpha\text{ Sin }\beta \;\ldots\ldots\; (2)$$

$$\text{Cos}(\alpha+\beta)=\text{Cos }\alpha\text{ Cos }\beta-\text{Sin }\alpha\text{ Sin }\beta \;\ldots\ldots\; (3)$$

$$\text{Cos}(\alpha-\beta)=\text{Cos }\alpha\text{ Cos }\beta+\text{Sin }\alpha\text{ Sin }\beta \;\ldots\ldots\; (4)$$

Adding (1) and (2), we get

$$\text{Sin}(\alpha+\beta)+\text{Sin}(\alpha+\beta)=2\text{Sin }\alpha\text{ Cos }\beta \qquad \ldots\ldots\; (5)$$

Subtracting (2) from (1)

$$\text{Sin}(\alpha+\beta)-\text{Sin}(\alpha-\beta)=2\text{Cos }\alpha\text{ Sin }\beta \qquad \ldots\ldots\; (6)$$

Adding (3) and (4), we have

$$\text{Cos}(\alpha+\beta)+\text{Cos}(\alpha-\beta)=2\text{Cos }\alpha\text{ Cos}\beta \qquad \ldots\ldots\; (7)$$

Subtracting (4) from (3), we have

$$\text{Cos}(\alpha+\beta)-\text{Cos}(\alpha-\beta)=-2\text{Sin }\alpha\text{ Sin }\beta \qquad \ldots\ldots\; (8)$$

With the help of (5), (6), (7) and (8), we have get another set of important formulas

Let $\alpha+\beta=\text{A}$ and $\alpha-\beta=\text{B}$

Adding these, we have

$$2\,\alpha=\text{A}+\text{B} \qquad \Rightarrow \alpha=\frac{\text{A}+\text{B}}{2}$$

Subtracting these, we have

$$2\,\beta=\text{A}-\text{B} \qquad \Rightarrow \beta=\frac{\text{A}-\text{B}}{2}$$

Now putting these values of α and β in formulas from (5) to (8), we get

$$\text{Sin A}+\text{Sin B}=2\text{ Sin}\frac{\text{A}+\text{B}}{2}\text{Cos}\frac{\text{A}-\text{B}}{2} \qquad (9)$$

$$\text{Sin A}-\text{Sin B}=2\text{ Cos}\frac{\text{A}+\text{B}}{2}\text{Sin}\frac{\text{A}-\text{B}}{2} \qquad (10)$$

$$\text{Cos A}+\text{Cos B}=2\text{ Cos}\frac{\text{A}+\text{B}}{2}\text{Cos}\frac{\text{A}-\text{B}}{2} \qquad (11)$$

$$\text{Cos A}-\text{Cos B}=-2\text{ Sin}\frac{\text{A}+\text{B}}{2}\text{Sin}\frac{\text{A}-\text{B}}{2} \qquad (12)$$

6.9 Converting Products to Sum or Difference:

If we write the formulas given in (5) to (8) in reverse order, we have

$$2\text{ Sin }\alpha\text{ Cos }\beta=\text{ Sin}(\alpha+\beta)+\text{Sin}(\alpha-\beta) \qquad (13)$$

$$2\ \text{Cos}\ \alpha\ \text{Sin}\ \beta = \text{Sin}(\alpha+\beta) - \text{Sin}(\alpha-\beta) \qquad (14)$$

$$2\ \text{Cos}\ \alpha\ \text{Cos}\ \beta = \text{Cos}(\alpha+\beta) + \text{Cos}(\alpha-\beta) \qquad (15)$$

$$2\ \text{Sin}\ \alpha\ \text{Sin}\ \beta = \text{Cos}(\alpha-\beta) - \text{Cos}(\alpha+\beta) \qquad (16)$$

The formulas from (13) to (16) express products into sum or difference.

Example 1:

Express $\text{Sin}8\,\theta + \text{Sin}4\,\theta$ as products.

Solution:

We use the formula

$$\text{Sin A} + \text{Sin B} = 2\ \text{Sin}\ \frac{A+B}{2}\ \text{Cos}\frac{A-B}{2}$$

$$\text{Sin}8\,\theta\ \text{Sin}4\,\theta = 2\ \text{Sin}\frac{8\theta+4\theta}{2}\ \text{Cos}\frac{8\theta-4\theta}{2}$$

$$= 2\ \text{Sin}\frac{12\,\theta}{2}\ \text{Cos}\frac{4\,\theta}{2}$$

$$\text{Sin}8\,\theta\ \text{Sin}4\,\theta = 2\ \text{Sin}6\,\theta\ .\ \text{Cos}2\,\theta$$

Example 2:

Express $\text{Sin}\ \theta + \text{Sin}3\ \theta + \text{Sin}5\ \theta + \text{Sin}7\ \theta$ as a product.

Solution:

$$\text{Sin}\ \theta + \text{Sin}3\ \theta + \text{Sin}5\ \theta + \text{Sin}7\ \theta$$

$$= (\text{Sin}7\ \theta + \text{Sin}\ \theta) + (\text{Sin}5\ \theta + \text{Sin}3\ \theta)$$

$$= 2\ \text{Sin}\left(\frac{7\theta+\theta}{2}\right)\ \text{Cos}\left(\frac{7\theta-\theta}{2}\right) + 2\text{Sin}\left(\frac{5\theta+3\theta}{2}\right)\text{Cos}\left(\frac{5\theta-3\theta}{2}\right)$$

$$= 2\ \text{Sin}\frac{8\,\theta}{2}\ \text{Cos}\frac{6\,\theta}{2} + 2\ \text{Sin}\frac{8\,\theta}{2}\text{Cos}\frac{2\,\theta}{2}$$

$$= 2\text{Sin}4\,\theta\ \text{Cos}\ 3\,\theta + 2\ \text{Sin}4\,\theta\ \text{Cos}\,\theta$$

$$= 2\text{Sin}4\,\theta\ [\text{Cos}\ 3\,\theta + \text{Cos}\,\theta]$$

$$= 2\ \text{Sin}4\,\theta\left[2\ \text{Cos}\frac{3\,\theta+\theta}{2}\text{Cos}\frac{3\,\theta-\theta}{2}\right]$$

$$= 2\ \text{Sin}4\,\theta\left[2\ \text{Cos}\frac{4\,\theta}{2}\text{Cos}\frac{2\,\theta}{2}\right]$$

$$= 2\ \text{Sin}4\,\theta\ [2\ \text{Cos}2\,\theta\ \text{Cos}\,\theta]$$

$$= 4\ \text{Sin}4\,\theta\ \text{Cos}2\,\theta\ \text{Cos}\,\theta$$

Example 3:

Prove that $\text{Sin}\ 10^{\circ}\ \text{Sin}\ 30^{\circ}\ \text{Sin}\ 50^{\circ}\ \text{Sin}\ 70^{\circ} = \frac{1}{16}$

Solution:

$$\text{L.H.S} = \text{Sin } 10^o \text{ Sin } 30^o \text{ Sin } 50^o \text{ Sin } 70^o$$

$$= \text{Sin } 30^o \text{ Sin } 10^o \text{ Sin } 50^o \text{ Sin } 70^o$$

$$= \frac{1}{2}[\text{Sin } 10^o \text{ Sin } 50^o] \text{ Sin } 70o \qquad \text{because Sin } 30^o = \frac{1}{2}$$

$$= \frac{1}{4}[2 \text{ Sin } 10^o \text{ Sin } 50^o] \text{ Sin } 70^o$$

Since, $2 \text{ Sin}\alpha \text{ Sin}\beta = \text{Cos}(\alpha - \beta) - \text{Cos}(\alpha + \beta)$

$$= \frac{1}{4}\left[\text{Cos}(10^o - 50^o) - \text{Cos}(10^o + 50^o)\right]\text{Sin}70^o$$

$$= \frac{1}{4}\left[\text{Cos}(-40^o) - \text{Cos } 60^o\right]\text{Sin}70^o$$

$$= \frac{1}{4}\left[\text{Cos}40^o - \frac{1}{2}\right]\text{Sin}70^o = \frac{1}{4}\left[\frac{2\text{Cos}40^o - 1}{2}\right]\text{Sin}70^o$$

$$= \frac{1}{8}\left[2\text{Sin}70^o \text{ Cos}40^o - \text{Sin}70^o\right]$$

We know $2 \text{ Sin}\alpha \text{ Cos}\beta = \text{Sin}(\alpha + \beta) + \text{Sin}(\alpha - \beta)$

$$= \frac{1}{8}\left[\text{Sin}(70^o + 40^o) + \text{Sin}(70^o - 40^o) - \text{Sin } 70^o\right]$$

$$= \frac{1}{8}\left[\text{Sin } 110^o + \text{Sin}30^o - \text{Sin}70^o)\right]$$

$$= \frac{1}{8}\left[\text{Sin}(180^o - 70^o) + \frac{1}{2} - \text{Sin}70^o\right]$$

$$= \frac{1}{8}\left[\text{Sin } 70^o + \frac{1}{2} - \text{Sin}70^o\right] = \frac{1}{8}\left(\frac{1}{2}\right) = \frac{1}{16} = \text{R.H.S}$$

Example 4:

Prove that $\dfrac{\text{Sin A} + 2\text{Sin}3\text{A} + \text{Sin}5\text{A}}{\text{Sin}3\text{A} + 2\text{Sin}5\text{A} + \text{Sin}7\text{A}} = \dfrac{\text{Sin}3\text{A}}{\text{Sin}5\text{A}}$

Solution:

$$\text{L.H.S} = \frac{\text{Sin A} + 2\text{Sin}3\text{A} + \text{Sin}5\text{A}}{\text{Sin}3\text{A} + 2\text{Sin}5\text{A} + \text{Sin}7\text{A}}$$

$$= \frac{(\text{Sin}5\text{A} + \text{Sin A}) + 2\text{Sin}3\text{A}}{(\text{Sin}7\text{A} + \text{Sin}3\text{A}) + 2\text{Sin}5\text{A}}$$

$$= \frac{2\text{Sin}\frac{5A+A}{2}\text{Cos}\frac{5A-A}{2}+2\text{Sin}3A}{2\text{Sin}\frac{7A+3A}{2}\text{Cos}\frac{7A-3A}{2}+2\text{Sin}5A}$$

$$= \frac{2\text{Sin}3A\ \text{Cos}\ 2A+2\text{Sin}3A}{2\text{Sin}5A\ \text{Cos}2A+2\text{Sin}5A}$$

$$= \frac{\mathbf{2Sin3A\ (Cos\ 2A+1)}}{\mathbf{2Sin5A\ (Cos2A+1)}}$$

$$= \frac{\text{Sin}3A}{\text{Sin}5A} = \text{R.H.S}$$

Exercise 6.3

Q.1 Express each of the following sum or difference as products.

(i) $\text{Sin}5\ \theta - \text{Sin}\ \theta$ (ii) $\text{Cos}\ \theta - \text{Cos}5\ \theta$

(iii) $\text{Cos}12\ \theta - \text{Cos}4\ \theta$ (iv) $\text{Sin}\frac{50}{3} - \text{Sin}\frac{50}{6}$

(v) $\text{Cos}\left(\frac{\alpha+\beta}{2}\right)+\text{Cos}\left(\frac{\alpha-\beta}{2}\right)$ (vi) $\text{Sin}4\ \theta + \text{Sin}2\ \theta$

Q.2 Express each of the following products as sum or difference.

(i) $2\ \text{Sin}3\ \theta\ \text{Cos}\ \theta$ (ii) $\text{Sin}3\ \theta\ .\ \text{Cos}5\ \theta$

(iii) $\text{Cos}3\ \theta\ .\ \text{Cos}5\ \theta$ (iv) $\text{Sin}\frac{\alpha+\beta}{2}\text{Cos}\frac{\alpha-\beta}{2}$

Q.3 Express $\text{Sin}3\ \theta + \text{Sin}5\ \theta + \text{Sin}7\ \theta + \text{Sin}9\ \theta$ as a product.

Prove the following identities:

Q.4 $\frac{\sin 5\theta - \sin 3\theta}{\cos 5\theta + \cos 3\theta} = \tan\theta$ Q5. $\frac{\sin 5\theta + \sin 3\theta}{\cos 5\theta - \cos 3\theta} = -\cot\theta$

Q.6 $\frac{\text{Cos}\ \beta + \text{Cos}\ 9\beta}{\text{Sin}\ \beta + \text{Sin}\ 9\beta} = \cot 5\beta$ Q.7 $\frac{\text{Sin}\ 3\theta - \text{Sin}\ \theta}{\text{Cos}^2\theta - \text{Sin}^2\theta} = 2\text{Sin}\ \theta$

Q.8 $\frac{\text{Sin A} + \text{Sin B}}{\text{Cos A} + \text{Cos B}} = \tan\frac{A+B}{2}$

Q.9 $\frac{\text{Cos}\ 2\theta - \text{Cos}\ 6\theta}{\text{Cos}\ 2\theta + \text{Cos}\ 6\theta} = \tan 4\theta \tan 2\theta$

Q.10 $\dfrac{\text{Cos } \alpha - \text{Cos } \beta}{\text{Cos } \alpha + \text{Cos } \beta} = -\dfrac{\tan\left(\dfrac{\alpha+\beta}{2}\right)}{\cot\left(\dfrac{\alpha-\beta}{2}\right)}$

Q.11 $\dfrac{\text{Sin } \theta + \text{Sin } 2\theta + \text{Sin } 3\theta}{\text{Cos } \theta + \text{Cos } 2\theta + \text{Cos } 3\theta} = \tan 2\theta$

Q.12 $\text{Sin } 5\theta + 2 \sin 3\theta + \sin \theta = 4 \sin 3\theta \cos^2\theta$

Q.13 Show that:

(i) $\dfrac{\text{Sin } 75^o - \text{Sin } 15^o}{\text{Cos } 75^o + \text{Cos } 15^o} = \dfrac{1}{\sqrt{3}}$

(ii) $\text{Sin } 20^o + \text{Sin } 40^o = \text{Cos } 10^o$

(iii) $\text{Cos } 80^o + \text{Cos } 40^o = \text{Cos}20^o$

(iv) $\text{Cos } 20^o + \text{Cos } 100^o + \text{Cos } 140^o = 0$

Prove that.

Q.14 $\text{Sin } 20^o \text{ Sin } 40^o \text{ Sin } 60^o \text{ Sin } 80^o = \dfrac{3}{16}$

Q.15 $\text{Sin } 10^o \text{ Sin } 30^o \text{ Sin } 50^o \text{ Sin } 70^o = \dfrac{1}{16}$

Q.16 $\text{Sin } 20^o \text{ Sin } 40^o \text{ Sin } 80^o \text{ Sin } 90^o = \dfrac{\sqrt{3}}{8}$

Q.17 $\text{Cos } 20^o \text{ Cos } 40^o \text{ Cos } 60^o \text{ Cos } 80^o = \dfrac{1}{16}$

Answer 6.3

Q.1 (i) $2 \text{ Cos } 3\theta \text{ Sin } 2\theta$ (ii) $2 \text{ Sin } 3\theta \text{ Sin } 2\theta$

(iii) $2 \text{ Cos } 8\theta \text{ Cos } 4\theta$ (iv) $2 \text{ Cos } \dfrac{150}{12} \text{ Sin } \dfrac{50}{12}$

(v) $2 \text{ Cos } \dfrac{\alpha}{2} \text{ Cos } \dfrac{\beta}{2}$ (vi) $2 \text{ Sin } 3\theta \text{ Cos } \theta$

Q.2 (i) $\text{Sin } 4\theta + \text{Sin } 2\theta$ (ii) $\dfrac{1}{2}[\text{Sin } 8\theta - \text{Sin } 2\theta]$

(iii) $\dfrac{1}{2}[\text{Cos } 8\theta + \text{Cos } 2\theta]$ (iv) $\dfrac{1}{2}[\text{Sin}\alpha + \text{Sin}\beta]$

Q.3 $4\text{Cos } \theta \text{ Sin } 6\theta \text{ Cos } 2\theta$