(iii) $\sqrt{41}\sin(\theta+\phi)$, $\phi = \tan^{-1}\left(\frac{4}{5}\right)$

(iv) $\sqrt{2}\sin(\theta+\phi)$, $\phi = \tan^{-1}(1)$

6.5 Double Angle Identities:

We know that:

$$\text{Sin}(\alpha+\beta) = \text{Sin}\alpha\ \text{Cos}\ \beta + \text{Cos}\alpha\ \text{Sin}\beta$$

Putting $\beta = \alpha$, **we have**

$$\boxed{\text{Sin}\ 2\alpha = 2\ \text{Sin}\alpha\ \text{Cos}\alpha}$$

$$\text{Sin}(\alpha+\alpha) = \text{Sin}\alpha\ \text{Cos}\alpha + \text{Cos}\alpha\ \text{Sin}\alpha$$

Also $\text{Cos}(\alpha+\alpha) = \text{Cos}\alpha\ \text{Cos}\beta - \text{Sin}\alpha\ \text{Sin}\beta$

Again putting $\beta = \alpha$ in this formula, we have

$$\text{Cos}(\alpha+\alpha) = \text{Cos}\alpha\ \text{Cos}\alpha - \text{Sin}\alpha\ \text{Sin}\alpha$$

$$\boxed{\text{Cos}2\alpha = \text{Cos}^2\alpha - \text{Sin}^2\alpha}$$

$$= \text{Cos}^2\alpha - (1-\text{Cos}^2\alpha) = \text{Cos}^2\alpha - 1 + \text{Cos}^2\alpha$$

$$\boxed{\text{Cos}\ 2\alpha = 2\text{Cos}^2\alpha - 1}$$

Again $\text{Cos}\ 2\alpha = \text{Cos}^2\alpha - \text{Sin}^2\alpha$

$$= 1 - \text{Sin}^2\alpha - \text{Sin}^2\alpha$$

$$\boxed{\text{Cos}\ 2\alpha = 1 - 2\ \text{Sin}^2\alpha}$$

Now, $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

Putting $\beta = \alpha$ in this formula, we have

$$\tan(\alpha+\alpha) = \frac{\tan\alpha + \tan\alpha}{1 - \tan\alpha\tan\alpha}$$

$$\boxed{\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}}$$

6.6 Half Angle identities:

We know that:

$$\text{Cos}\ 2\alpha = 1 - 2\ \text{Sin}^2\alpha$$

Therefore $$2\ \text{Sin}^2\alpha = 1 - \text{Cos}\ 2\alpha$$

$$\text{Sin}^2\alpha = \frac{1 - \text{Cos}\ 2\alpha}{2}$$

$$\text{Sin}\ \alpha = \pm\sqrt{\frac{1 - \text{Cos}\ 2\alpha}{2}}$$

Putting $2\alpha = \theta \Rightarrow \alpha = \frac{\theta}{2}$ in this formula

We have $$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1 - \cos 2(\frac{\theta}{2})}{2}}$$

$$\text{Sin}\ \frac{\theta}{2} = \pm\sqrt{\frac{1 - \cos\theta}{2}} \quad \ldots\ldots\ldots\text{(i)}$$

Also we know that

$$\text{Cos}\ 2\alpha = 2\ \text{Cos}^2\alpha - 1$$

$\therefore$ $$\text{Cos}^2\alpha = \frac{1 + \text{Cos}\ 2\alpha}{2}$$

Or $$\text{Cos}\ \alpha = \pm\sqrt{\frac{1 + \text{Cos}\ 2\alpha}{2}}$$

Put $2\alpha = \theta \Rightarrow \alpha = \frac{\theta}{2}$ in this formula we have

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1 + \cos 2(\frac{\theta}{2})}{2}}$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1 + \cos\theta)}{2}} \quad \ldots\ldots\ldots\text{(ii)}$$

Now , $$\tan\frac{\theta}{2} = \frac{\text{Sin}\frac{\theta}{2}}{\text{Cos}\frac{\theta}{2}}$$

From (i) and (ii) $= \dfrac{\pm\sqrt{\frac{1-\cos\theta}{2}}}{\pm\sqrt{\frac{1+\cos\theta}{2}}}$

$$\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\text{Cos }\theta}{1+\text{Cos }\theta}}$$

Example 1:

If $\text{Sin }\theta = \frac{4}{5}$ and the terminal ray of θ is in the second quadrant. Find the value of (i) $\text{Sin }2\theta$ (ii) $\text{Cos }\frac{\theta}{2}$

Solution:

Because $\text{Cos }\theta = \sqrt{1-\text{Sin}^2\theta}$

$$= \sqrt{1-\left(\frac{4}{5}\right)^2} = \sqrt{1-\frac{16}{25}}$$

$$= \sqrt{\frac{25-16}{25}} = \sqrt{\frac{9}{25}} = \pm\frac{3}{5}$$

$\text{Cos }\theta = -\frac{3}{5}$ because the terminal ray of θ is in 2nd quadrant.

(i) $\text{Sin }2\theta = 2\text{ Sin }\theta\text{ Cos }\theta$

$$= 2\left(\frac{4}{5}\right)\left(-\frac{3}{5}\right) = -\frac{24}{25}$$

(ii) $\text{Cos }\frac{\theta}{2} = \sqrt{\frac{1+\text{Cos }\theta}{2}} = \sqrt{\frac{1-\frac{3}{5}}{2}} = \sqrt{\frac{\frac{2}{5}}{2}}$

$$= \sqrt{\frac{2}{10}} = \frac{1}{\sqrt{5}}$$

6.7 Triple angle identities:

(i) $\text{Cos }3\theta = 4\text{ Cos}^3\theta - 3\text{ Cos }\theta$

(ii) $\text{Sin }3\theta = 3\sin\theta - 4\sin^3\theta$

(iii) $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$

Prove that :

(i) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

L.H.S. $= \cos 3\theta$

$= \cos(2\theta + \theta)$

$= \cos 2\theta \cos\theta - \sin 2\theta \sin\theta$

$= (2\cos^2\theta - 1)\cos\theta - 2\sin\theta\cos\theta\sin\theta$

$= 2\cos^3\theta - \cos\theta - 2\cos\theta(1 - \cos^2\theta)$

$= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta$

$= 4\cos^3\theta - 3\cos\theta$ = R.H.S.

(ii) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

L.H.S. $= \sin 3\theta$

$= \sin(2\theta + \theta)$

$= \sin 2\theta\cos\theta + \cos 2\theta\sin\theta$

$= 2\sin\theta\cos\theta\cos\theta + (1 - 2\sin^2\theta)\sin\theta$

$= 2\sin\theta - 2\sin^3\theta + \sin\theta - 2\sin^3\theta$

$= 3\sin\theta - 4\sin^3\theta$ = R.H.S.

(iii) $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

L.H.S. $= \tan 3\theta$

$= \tan(2\theta + \theta)$

$$= \frac{\tan 2\theta + \tan\theta}{1 - \tan 2\theta\tan\theta} = \frac{\dfrac{2\tan\theta}{1 - \tan^2\theta} + \tan\theta}{1 - \dfrac{2\tan\theta}{1 - \tan^2\theta}\tan\theta}$$

$(\because \tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta})$

$$= \frac{2\tan\theta + \tan\theta - \tan^3\theta}{1 - \tan^2\theta - 2\tan^2\theta}$$

$$= \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} \qquad = \text{R.H.S.}$$

Example 2:

Show that $\text{Cosec}\,2\theta - \text{Cot}\,2\theta = \tan\theta$

Solution:

$$\text{L.H.S} = \text{Cosec}\,2\theta - \text{Cot}\,2\theta$$

$$= \frac{1}{\text{Sin}\,2\theta} - \frac{\text{Cos}\,2\theta}{\text{Sin}\,2\theta}$$

$$= \frac{1 - \text{Cos}\,2\theta}{\text{Sin}\,2\theta}$$

$$= \frac{1-(1-2\sin^2\theta)}{2\sin\theta\cos\theta} \qquad \because\ \text{Cos}\,2\theta = 1 - 2\,\text{Sin}^2\theta$$

$$= \frac{2\,\text{Sin}^2\theta}{2\,\text{Sin}\,\theta\,\text{Cos}\,\theta} = \frac{\text{Sin}\,\theta}{\text{Cos}\,\theta}$$

$$\because\ \text{Sin}\,2\theta = 2\,\text{Sin}\,\theta\,\text{Cos}\,\theta$$

$$= \tan\theta = \text{R.H.S}$$

Example 3:

Using Half angle formula find

(i) $\text{Sin}\,210^o$ (ii) $\text{Cos}\,210^o$ (iii) $\tan 210^0$

Solution:

(i) $$\text{Sin}\frac{\theta}{2} = \pm\sqrt{\frac{1-\text{Cos}\,\theta}{2}}$$

$$\text{Sin}\,210^o = \pm\sqrt{\frac{1-\text{Cos}\,420^o}{2}} = \pm\sqrt{\frac{1-\text{Cos}\,60^o}{2}}$$

$$= \pm\sqrt{\frac{1-\frac{1}{2}}{2}} = \pm\sqrt{\frac{\frac{1}{2}}{2}} = \pm\sqrt{\frac{1}{4}}$$

$$\text{Sin}\,210^o = \pm\frac{1}{2}$$

(ii) $$\text{Cos}\frac{\theta}{2} = \pm\sqrt{\frac{1+\text{Cos}\,\theta}{2}}$$

$$\text{Cos}\,210^o = \pm\sqrt{\frac{1+\text{Cos}\,420}{2}} = \pm\sqrt{\frac{1+\text{Cos}\,60^o}{2}}$$

$$= \pm\sqrt{\frac{1+\frac{1}{2}}{2}} = \pm\sqrt{\frac{\frac{3}{2}}{2}} = \pm\frac{\sqrt{3}}{\sqrt{4}}$$

$$\text{Cos } 210^{o} = \pm\frac{\sqrt{3}}{2}$$

(iii) $$\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \sqrt{\frac{1-\cos 420^{o}}{1+\cos 420^{o}}} \quad (\theta = 420^{o})$$

$$\tan\frac{420^{o}}{2} = \sqrt{\frac{1-\cos 60^{o}}{1+\cos 60^{o}}} = \sqrt{\frac{1-\frac{1}{2}}{1+\frac{1}{2}}}$$

$$\tan 210^{o} = \sqrt{\frac{\frac{2-1}{2}}{\frac{2+1}{2}}} = \sqrt{\frac{1}{2}\times\frac{2}{3}} = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}}$$

Example 4:

Prove that $\frac{\text{Sin } 2\text{ A}}{\text{Sin A}} - \frac{\text{Cos } 2\text{ A}}{\text{Cos A}} = \text{Sec A}$

Solution:

$$\text{L.H.S} = \frac{\text{Sin } 2\text{ A}}{\text{Sin A}} - \frac{\text{Cos } 2\text{ A}}{\text{Cos A}}$$

$$= \frac{\text{Sin } 2\text{ A Cos A} - \text{Cos } 2\text{ A Sin A}}{\text{Sin A Cos A}}$$

$$= \frac{\text{Sin } (2\text{A} - \text{A})}{\text{Sin A Cos A}} = \frac{\text{Sin A}}{\text{Sin A Cos A}}$$

$$= \frac{1}{\text{Cos A}}$$

$$= \text{Sec A}$$

Exercise 6.2

Q.1 If $\text{Cos}\,\theta = \frac{4}{5}$ and the terminal ray of θ is in the first quadrant find the value of

(i) $\text{Sin}\frac{\theta}{2}$ (ii) $\text{Cos}\frac{\theta}{2}$ (iii) $\tan\frac{\theta}{2}$

Q.2 If $\text{Sin}\,\theta = \frac{4}{5}$ and the terminal ray of θ is in the first quadrant, find the value of

(i) $\text{Sin}\,2\theta$ (ii) $\text{Cos}\,2\theta$

Q.3 If $\text{Cos}\,\theta = -\frac{5}{13}$ and the terminal side of θ is in the second quadrant, find the value of

(i) $\text{Sin}\frac{\theta}{2}$ (ii) $\text{Cos}\frac{\theta}{2}$

Q.4 If $\tan\,\theta = -\frac{1}{5}$, the terminal ray of θ lies in the second quadrant, then find:

(i) $\text{Sin}\,2\,\theta$ (ii) $\text{Cos}\,2\,\theta$

Prove the following identities:

Q.5 $\cos\theta = 2\cos^2\frac{\theta}{2} - 1$

Q.6 $\sin 2\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}$

Q.7 $\cos 2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$

Q.8 $\tan\theta = \dfrac{\sin 2\theta}{1+\cos 2\theta}$

Q.9 $\cot\theta = \dfrac{\sin 2\theta}{1-\cos 2\theta}$

Q.10 $\dfrac{1+\sin 2A - \cos 2A}{1+\sin 2A + \cos 2A} = \tan A$

Q.11 $\sec 2A + \tan 2A = \dfrac{\cos A + \sin A}{\cos A - \sin A}$

Q.12 $\text{cosec}\,2\theta + \cot 2\theta = \cot\theta$

Q.13 $\dfrac{1-\cos 2A}{1+\cos 2A} = \tan^2 A$

Q.14 $\dfrac{\sin 3\theta}{\cos\theta} + \dfrac{\cos 3\theta}{\sin\theta} = 2\cot 2\theta$

Q.15 $\dfrac{\cot^2\theta - 1}{\operatorname{cosec}^2\theta} = \cos 2\theta$

Q.16 $\cos^4\theta - \sin^4\theta = \dfrac{1}{\sec 2\theta}$

Q.17 $\left(\sin\dfrac{\theta}{2} + \cos\dfrac{\theta}{2}\right)^2 = 1 + \sin\theta$

Q.18 $(\sin\theta - \cos\theta)^2 = 1 - \sin 2\theta$

Q.19 $\sin^4\theta = \dfrac{3}{8} - \dfrac{1}{2}\cos 2\theta + \dfrac{1}{8}\cos 4\theta$

Q.20 Compute the value of $\sin\dfrac{\pi}{12}$ from the function of $\dfrac{\pi}{6}$

Answers 6.2

Q.1 (i) $\dfrac{1}{\sqrt{10}}$ (ii) $\dfrac{3}{\sqrt{10}}$ (iii) $\dfrac{1}{3}$

Q.2 (i) $\dfrac{24}{25}$ (ii) $-\dfrac{7}{25}$

Q.3 (i) $\dfrac{3}{\sqrt{13}}$ (ii) $\dfrac{2}{\sqrt{13}}$

Q.4 (i) $-\dfrac{5}{13}$ (ii) $\dfrac{12}{13}$

Q.21 $\dfrac{\sqrt{2-\sqrt{3}}}{2}$