5.13 Fundamental Identities:

For any real number θ, we shall derive the following three fundamental identities

(i) $\text{Cos}^2\theta + \text{Sin}^2\theta = 1$
(ii) $\text{Sec}^2\theta = 1 + \tan^2\theta$
(iii) $\text{Cosec}^2\theta = 1 + \text{Cot}^2\theta$

Proof :

Consider an angle $\angle XOP = \theta$ in the standard position. Take a point P on the terminal line of the angle θ. Draw PQ perpendicular from P on OX.

From fig., ΔOPQ is a right angled triangle. By pythagoruse theorem

$$(OP)^2 = (OQ)^2 + (PQ)^2$$

Or, $z^2 = x^2 + y^2$

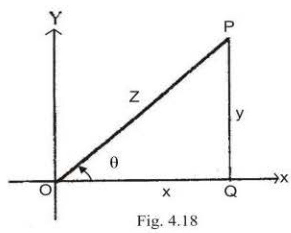

Fig. 4.18

(i) Dividing both sides by z^2

then $$\frac{z^2}{z^2} = \frac{x^2}{z^2} + \frac{y^2}{z^2}$$

$$1 = \left(\frac{x}{z}\right)^2 + \left(\frac{y}{z}\right)^2$$

$$1 = (\text{Cos }\theta)^2 + (\text{Sin }\theta)^2$$

$$1 = \text{Cos}^2\theta + \text{Sin}^2\theta$$

or, $$\text{Cos}^2\theta + \text{Sin}^2\theta = 1$$

(ii) Dividing both sides of Eq. (i) by x^2, we have

$$\frac{z^2}{x^2} = \frac{x^2}{x^2} + \frac{y^2}{x^2}$$

$$\left(\frac{z}{x}\right)^2 = 1 + \left(\frac{y}{x}\right)^2$$

$$(\text{Sec }\theta)^2 = 1 + (\tan\theta)^2$$

$$\text{Sec}^2\theta = 1 + \tan^2\theta$$

(iii) Again, dividing both sides of Eq (i) by y^2, we have

$$\frac{z^2}{y^2} = \frac{x^2}{y^2} + \frac{y^2}{y^2}$$

$$\left(\frac{z}{y}\right)^2 = \left(\frac{x}{y}\right)^2 + 1$$

$$(\text{Cosec } \theta)^2 = (\text{Cot } \theta)^2 + 1$$

$$\text{Cosec}^2 \theta = \text{Cot}^2 \theta + 1$$

$$\text{Cosec}^2 \theta = 1 + \text{Cot}^2 \theta$$

Example 1:

Prove that $\frac{\text{Sin x}}{\text{Cosec x}} + \frac{\text{Cos x}}{\text{sec x}} = 1$

Solution:

$$\text{L.S.H.} = \frac{\text{Sin x}}{\text{Cosec x}} + \frac{\text{Cos x}}{\text{sec x}}$$

$$= \text{Sin x}.\frac{1}{\text{Cosec x}} + \text{Cos x}.\frac{1}{\text{sec x}} \quad \because \quad \frac{1}{\text{Cosec x}} = \text{Sin x}$$

$$= \text{Sin x. Sin x} + \text{Cos x. Cos x} \quad \because \quad \frac{1}{\text{Sec x}} = \text{Cos x}$$

$$= \text{Sin}^2 \text{ x} + \text{Cos}^2 \text{ x}$$

$$= 1$$

$$= \text{R.H.S}$$

Example 2:

Prove that $\frac{\text{Sec x} - \text{Cos x}}{1 + \text{Cos x}} = \text{Sec x} - 1$

Solution:

$$\text{L.H.S} = \frac{\text{Sec x} - \text{Cos x}}{1 + \text{Cos x}}$$

$$= \frac{\frac{1}{\text{Cos x}} - \text{Cos x}}{1 + \text{Cos x}}$$

$$= \frac{\frac{1 - \text{Cos}^2 \text{ x}}{\text{Cos x}}}{1 + \text{Cos x}} = \frac{1 - \text{Cos}^2 \text{ x}}{\text{Cos x}(1 + \text{Cos x})}$$

$$= \frac{(1 - \text{Cos x})(1 + \text{Cos x})}{\text{Cos x}(1 + \text{Cos x})}$$

$$= \frac{1 - \text{Cos x}}{\text{Cos x}} = \frac{1}{\text{Cos x}} - \frac{\text{Cos x}}{\text{Cos x}}$$

$$= \text{Sex x} - 1 = \text{R.H.S.}$$

Example 3:

prove that $\sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}} = \sec\theta - \tan\theta$

Solution: L.H.S. $= \sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}}$

$$= \sqrt{\frac{(1-\sin\theta)(1-\sin\theta)}{(1+\sin\theta)(1-\sin\theta)}}$$

$$= \sqrt{\frac{(1-\sin\theta)^2}{1-\sin^2\theta}} \qquad = \sqrt{\frac{(1-\sin\theta)^2}{\cos^2\theta}}$$

$$= \frac{(1-\sin\theta)}{\cos\theta} \qquad = \frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta}$$

$$= \sec\theta - \tan\theta \qquad = \text{R.H.S.}$$

Exercise 5.3

Prove the following Identities:

Q.1 $1 - 2\,\text{Sin}^2\,\theta = 2\,\text{Cos}^2\,\theta - 1$

Q.2 $\text{Cos}^4\,\theta - \text{Sin}^4\,\theta = 1 - 2\,\text{Sin}^2\theta$

Q.3 $\dfrac{1}{\text{Cosec}^2\,\theta} + \dfrac{1}{\text{Sec}^2\,\theta} = 1$

Q.4 $\dfrac{1}{\tan\theta + \text{Cot}\,\theta} = \text{Sin}\,\theta.\text{Cos}\,\theta$

Q.5 $(\text{Sec}\,\theta - \tan\theta)^2 = \dfrac{1-\text{Sin}\,\theta}{1+\text{Sin}\,\theta}$

Q.6 $(\text{Cosec}\,\theta - \text{Cot}\,\theta)^2 = \dfrac{1-\text{Cos}\,\theta}{1+\text{Cos}\,\theta}$

Q.7 $(1-\text{Sin}^2\theta)(1+\tan^2\theta)=1$

Q.8 $\dfrac{1}{1+\text{Sin}\,\theta} + \dfrac{1}{1-\text{Sin}\,\theta} = 2\text{Sec}^2\theta$

Q.9 $\sqrt{\dfrac{1-\text{Sin}\,\theta}{1+\text{Sin}\,\theta}} = \text{Sec}\theta - \tan\theta$

Q.10 $\sqrt{\dfrac{1+\text{Cos}\,\theta}{1-\text{Cos}\,\theta}} = \text{Cosec}\,\theta + \cot\theta$

Q.11 $\dfrac{1-\tan A}{1+\tan A}=\dfrac{\text{Cot } A-1}{\text{Cot } A+1}$

Q.12 $\dfrac{\text{Cot}^2\theta-1}{\text{Cot}^2\theta+1}=2\text{Cos}^2\theta-1$

Q.13 $\dfrac{\tan\theta}{1-\text{Cot }\theta}+\dfrac{\text{Cot }\theta}{1-\tan\theta}=\text{Sec }\theta\text{ Cosec }\theta+1$

Q.14 $\dfrac{\text{Sec }\theta-\tan\theta}{\text{Sec }\theta+\tan\theta}=1-2\text{Sec }\theta\tan\theta+2\tan^2\theta$

Q.15 $\dfrac{1+\tan^2\theta}{1+\cot^2\theta}=\dfrac{(1-\tan\theta)^2}{(1-\cot\theta)^2}$

Q.16 $\text{Cosec } A+\text{Cot } A=\dfrac{1}{\text{Cosec } A-\text{Cot } A}$

Q.17 $\dfrac{1}{\text{Sec }\theta+\tan\theta}=\dfrac{1-\text{Sin }\theta}{\text{Cos }\theta}=\text{Sec } x-\tan x$

Q.18 $(1-\tan\theta)^2+(1-\text{Cot }\theta)^2=(\text{Sec }\theta-\text{Cosec }\theta)^2$

Q.19 $\dfrac{\text{Cos}^3 t-\text{Sin}^3 t}{\text{Cos } t-\text{Sin } t}=1+\text{Sin } t\text{ Cos } t$

Q.20 $\text{Sec}^2 A+\tan^2 A=(1-\text{Sin}^4 A)\text{ Sec}^4 A$

Q.21 $\dfrac{\text{Sec } x-\text{Cos } x}{1+\text{Cos } x}=\text{Sec } x-1$

Q.22 $\dfrac{1+\text{Sin }\theta+\text{Cos }\theta}{1+\text{Sin }\theta-\text{Cos }\theta}=\dfrac{\text{Sin }\theta}{1-\text{Cos }\theta}$

Q.23 $\dfrac{\text{Sin } x+\text{Cos } x}{\tan^2 x-1}=\dfrac{\text{Cos}^2 x}{\text{Sin } x-\text{Cos } x}$

Q.24 $(1+\text{Sin }\theta)(1-\text{Sin }\theta)=\dfrac{1}{\text{Sec}^2\theta}$

Q.25 $\dfrac{\tan\theta}{\text{Sec }\theta-1}+\dfrac{\tan\theta}{\text{Sec }\theta+1}=2\text{Cosec }\theta$

Q.26 $\dfrac{\cot\theta\cos\theta}{\cot\theta+\cos\theta}=\dfrac{\cot\theta-\cos\theta}{\cot\theta\cos\theta}$

Q.27 If $m=\tan\theta+\text{Sin }\theta$ and $n=\tan\theta-\text{Sin }\theta$ than prove that $m^2-n^2=4\sqrt{m\,n}$