Chapter 5
Fundamentals of Trigonometry

5.1 Introduction:

The word "trigonometry" is a Greek word. Its mean "measurement of a triangle". Therefore trigonometry is that branch of mathematics concerned with the measurement of sides and angle of a plane triangle and the investigations of the various relations which exist among them. Today the subject of trigonometry also includes another distinct branch which concerns itself with properties relations between and behavior of trigonometric functions.

The importance of trigonometry will be immediately realized when its applications in solving problem of mensuration, mechanics physics, surveying and astronomy are encountered.

5.2 Types of Trigonometry:

There are two types of trigonometry

(1) Plane Trigonometry (2) Spherical Trigonometry

1. Plane Trigonometry

Plane trigonometry is concerned with angles, triangles and other figures which lie in a plane.

2. Spherical Trigonometry

Spherical Trigonometry is concerned with the spherical triangles, that is, triangles lies on a sphere and sides of which are circular arcs.

5.3 Angle:

An angle is defined as the union of two non-collinear rays which have a common end-points.

An angle is also defined as it measures the rotation of a line from one position to another about a fixed point on it.

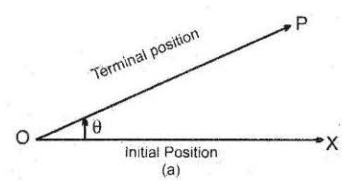


Fig. 4.1

In figure 5.1(a)the first position OX is called initial line (position)and second position OP is called terminal line or generating line(position) of $\angle XOP$.

If the terminal side resolves in anticlockwise direction the angle described is positive as shown in figure (i)

If terminal side resolves in clockwise direction, the angle described is negative as shown in figure (ii)

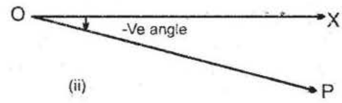


Fig. 4.3

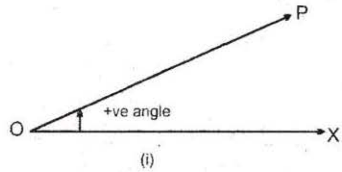


Fig. 4.2

5.4 Quadrants:

Two mutually perpendiculars straight lines xox□ and yoy□ divide the plane into four equal parts, each part is called quadrant.
Thus XOY, X′OY, X′OY′ and XOY′ are called the Ist, IInd, IIIrd and IVth quadrants respectively.

In first quadrant the angle vary from 0^o to 90^o in anti-clockwise direction and from -270^o to -360^o in clockwise direction.

In second quadrant the angle vary from 90^o to 180^o in anti-clockwise direction and -180^o to -270^o in clockwise direction.

In third quadrant the angle vary from 180^o to 270^o in anti-clockwise direction and from -90^o to -180^o in clockwise direction.

In fourth quadrant the angle vary from 270^o to 360^o in anti-clockwise direction and from -0^o to -90^o in clockwise direction.

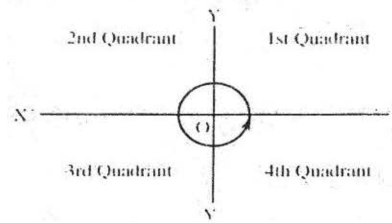


Fig. 4.4

5.5 Measurement of Angles:

The size of any angle is determined by the amount of rotations. In trigonometry two systems of measuring angles are used.

(i) Sexagesimal or English system (Degree)

(ii) Circular measure system (Radian)

(i) Sexagesimal or English System (Degree)

The sexagesimal system is older and is more commonly used. The name derive from the Latin for "sixty". The fundamental unit of angle measure in the sexagesimal system is the degree of arc. By definition, when a circle is divided into 360 equal parts, then

$$\text{One degree} = \frac{1}{360}\text{th part of a circle.}$$

Therefore, one full circle = 360 degrees.

The symbol of degrees is denoted by $(\ )^0$.

Thus an angle of 20 degrees may be written as 20^{o}.

Since there are four right angles in a complete circle .

One right angle = $\frac{1}{4}$ circle = $\frac{1}{4}(360^{o}) = 90^{o}$

The degree is further subdivided in two ways, depending upon whether we work in the common sexagesimal system or the decimal sexagesimal system. In the common sexagesimal system, the degree is subdivided into 60 equal parts, called minutes, denoted by the symbol$(\ )'$, and the minute is further subdivided into 60 equal parts, called second, indicated by the symbol $(\ )''$. Therefore

1 minute = 60 seconds

1 degree = 60 minutes = 3600 seconds

1 circle = 360 degrees = 21600 minutes = 12,96,000 sec.

In the decimal sexagesimal system, angles smaller than 1^{o} are expressed as decimal fractions of a degree. Thus one-tenth $\left(\frac{1}{10}\right)$ of a degree is expressed as 0.1^{o} in the decimal sexagesimal system and as $6'$ in the common sexagesimal system; one-hundredth $\left(\frac{1}{100}\right)$ of a degree is 0.01^{o} in the decimal system and 36" in the common system; and $47\frac{1}{9}$ degrees comes out $(47.111 \ldots)^{o}$ in the decimal system and $47^{o}6'40''$ in the common system.

(ii) Circular measure system (Radian)

This system is comparatively recent. The unit used in this system is called a Radian. The Radian is define "The angle subtended at the centre of a circle by an arc equal in length to the radius of the circle." As shown in fig., Arc AB is equal in length to the radius $\overline{OB}$ of the circle. The subtended, $\angle AOB$ is then one radian.

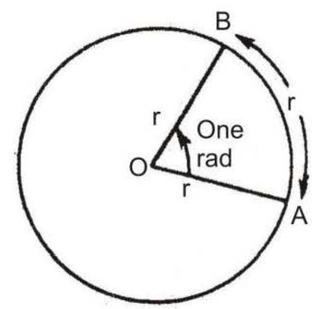


Fig. 4.5

i.e. m $\angle AOB = 1$ radian.

5.6 Relation between Degree and Radian Measure:

Consider a circle of radius r, then the circumference of the circle is $2\pi r$. By definition of radian,

An arc of length 'r' subtends an angle = 1 radian

$\therefore$ An arc of length $2\pi r$ subtends an angle = 2π radian

Also an arc of length $2\pi r$ subtends an angle = 360^{o}

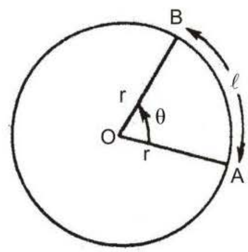


Fig. 4.6

Then 2π radians = 360^o

Or π radians = 180^o

$$1 \text{ radians} = \frac{180^o}{\pi}$$

$$1 \text{ radians} = \frac{180}{3.1416}$$

Or 1 radians = 57.3^o

Therefore to convert radians into degree,

we multiply the number of radians by $\frac{180^o}{\pi}$ or 57.3.

Now Again

$$360^o = 2\pi \text{ radians}$$

$$1^o = \frac{2\pi r}{360^o} \text{ radians}$$

$$1^o = \frac{\pi}{180^o}$$

Or $$1^o = \frac{3.1416}{180^o}$$

1^o = 0.01745 radians

Therefore, to convert degree into radians, we multiply the number of degrees by $\frac{\pi}{180}$ or 0.0175.

Note: One complete revolution = 360^o = 2π radius.

5.7 Relation between Length of a Circular Arc and the Radian Measure of its Central Angle:

Let "l" be the length of a circular arc $\overline{AB}$ of a circle of radius r, and θ be its central angle measure in radians. Then the ratio of l to the circumference $2\pi r$ of the circle is the same as the ratio of θ to 2π.

Therefore

$$l : 2\pi r = \theta : 2\pi$$

Or $$\frac{l}{2\pi r} = \frac{\theta}{2\pi}$$

$$\frac{l}{r} = \theta$$

$l = \theta r$, where θ is in radian

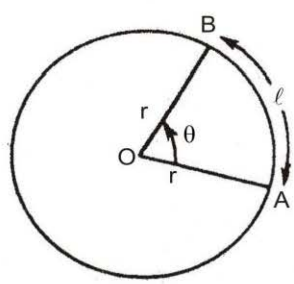


Fig. 4.7

Fig. 5.7

Note:

If the angle will be given in degree measure we have to convert it into Radian measure before applying the formula.

Example 1:

Convert 120^o into Radian Measure.

Solution:

120^o

$$120^o = 120 \text{ x } \frac{\pi}{180}$$

$$= \frac{2\pi}{3} = \frac{2(3.1416)}{3} = 2.09 \text{ rad.}$$

Example 2:

Convert $37^o\ 25'\ 38''$ into Radian measure.

Solution:

$$37^o\ 25'\ 28'' = 37^o + \frac{25}{60} + \frac{28}{3600}$$

$$= 37^o + \frac{5^o}{12} + \frac{7^o}{900}$$

$$= 37^o + \frac{382^o}{900}$$

$$= 37 + \frac{181^o}{450} = \frac{16831^o}{450} = \frac{16831}{450} \text{x} \frac{\pi}{180}$$

$$= \frac{16831(3.14160)}{81000} = \frac{52876.26}{81000}$$

$$37^o\ 25'\ 28'' = 0.65 \text{ radians}$$

Example 3:

Express in Degrees:

(i) $\frac{5\pi}{3}$ rad (ii) 2.5793 rad (iii) $\frac{\pi}{6}$ rad (iv) $\frac{\pi}{3}$ rad

Solution:

(i) Since 1 rad $= \frac{180}{\pi}$ deg

$$\therefore \frac{5\pi}{3} \text{ rad} = \frac{5\pi}{3} \text{ x } \frac{180}{\pi} \text{ deg} = 5(60) \text{ deg} = 300^o$$

(ii) 2.5793 rad $= 2.5793\ (57^o .29578)$ $\therefore$ 1 rad $= 57^o .295778$

$= 147^o.78301 = 147.78$ (two decimal places)

(iii) $\frac{\pi}{6}$ rad $= \frac{\pi}{6} \times \frac{12}{\pi}$ deg $= 30^o$

(iv) $\frac{\pi}{3}$ rad $= \frac{\pi}{3} \times \frac{12}{\pi}$ deg $= 60^o$

Example 4:

What is the length of an arc of a circle of radius 5 cm. whose central angle is of 140^o?

Solution: l = length of an arc = ?

r = radius = 5 cm

$\theta = 140^o$

Since 1 deg = 0.01745 rad

$\therefore \theta = 140 \times 0.01745$ rad $= 2.443$ rad

$\therefore l = r\theta$

$l = (5)(2.443) = 12.215$ cm

Example 5:

A curve on a highway is laid out an arc of a circle of radius 620m. How long is the arc that subtends a central angle of 32^o?

Solution: $r = 620$m $l = ?$ $\theta = 32^o = 32 \times \frac{\pi}{180}$ rad

$$l = 620 \times 32 \times \frac{\pi}{180} = 346.41 \text{ m}$$

Example 6:

A railway Train is traveling on a curve of half a kilometer radius at the rate of 20 km per hour through what angle had it turned in 10 seconds?

Solution:

Radius $= r = \frac{1}{2}$ km, $\theta = ?$

We know $s = vt$

v = velocity of Train = 20km/hour $= \frac{20}{3600}$ km/sec.

$$v = \frac{1}{180} \text{ km/sec}$$

l = Distance traveled by train in 10 seconds $= \frac{1}{180} \times 10$ km/sec

$$l = \frac{1}{18}$$

Since $l = r\theta$

$\Rightarrow \quad \frac{1}{18} = \frac{1}{2}\theta$

$\theta = \frac{2}{18} = \frac{1}{9}\text{rad}$

Example7

The moon subtends an angle of 0.5^o as observed from the Earth. Its distance from the earth is 384400 km. Find the length of the diameter of the Moon.

Solution:

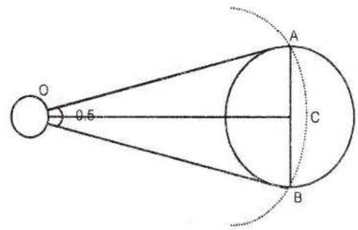


Fig. 4.8

l = AB = diameter of the Moon = ?

as angle 0.5 is very small.

i.e. AB (arc length) consider as a straight line AB

θ = 0.5^o = 0.5 x 0.01745 rad = 0.008725 rad

r = OC = d = distance between the earth and the moon

r = OC = 3844000 km

Since $l = r\theta$

l= 384400 x 0.008725 = 3353.89 km

Exercise 5.1

1. Convert the following to Radian measure
   (i) 210^o (ii) 540^o (iii) $42^o\ 36'\ 12''$ (iv) $24^o32'30''$
2. Convert the following to degree measure:
   (i) $\frac{5\pi}{4}$ rad (ii) $\frac{2\pi}{3}$ rad (iii) 5.52 rad (iv) 1.30 rad
3. Find the missing element l, r, θ when:
   (i) l = 8.4 cm, θ = 2.8 rad
   (ii) l = 12.2 cm, r = 5cm
   (iii) r = 620m, $\theta = 32^o$
4. How far a part are two cities on the equator whose longitudes are 10^oE and 50^oW? (Radius of the Earth is 6400km)

5. A space man land on the moon and observes that the Earth's diameter subtends an angle of $1^\circ\ 54'$ at his place of landing. If the Earth's radius is 6400km, find the distance between the Earth and the Moon.
6. The sun is about 1.496×10^{8} km away from the Earth. If the angle subtended by the sun on the surface of the earth is 9.3×10^{-3} radians approximately. What is the diameter of the sun?
7. A horse moves in a circle, at one end of a rope 27cm long, the other end being fixed. How far does the horse move when the rope traces an angle of 70° at the centre.
8. Lahore is 68km from Gujranwala. Find the angle subtended at the centre of the earth by the road. Joining these two cities, earth being regarded as a sphere of 6400km radius.
9. A circular wire of radius 6 cm is cut straightened and then bend so as to lie along the circumference of a hoop of radius 24 cm. find measure of the angle which it subtend at the centre of the hoop
10. A pendulum 5 meters long swings through an angle of 4.5^0. through what distance does the bob moves ?
11. A flywheel rotates at 300 rev/min. If the radius is 6 cm. through what total distance does a point on the rim travel in 30 seconds ?

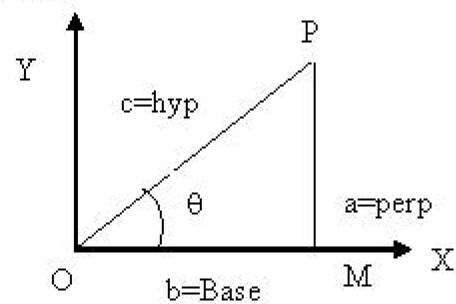
Answers 5.1

1	(i) 3.66 rad	(ii) 3π	(iii) 0.74 rad	(iv) 0.42 rad
2.	(i) 225°	(ii) 120°	(iii) $316^\circ\ 16'\ 19''$	(iv) $74^\circ\ 29'\ 4''$
3.	(i) r = 3cm	(ii) $\theta = 2.443$ rad	(iii) $l = 346.4$ meters	
4.	6704.76 km	5. 386240 km		
6.	1.39×10^{6} km	7. 33 m	8. $36'\ 43''$	9. $\pi/2$ or 90^0
10.	0.39 m	11. 5657 cm		

5.8 Trigonometric Function and Ratios:

Let the initial line OX revolves and trace out an angle θ. Take a point P on the final line. Draw perpendicular PM from P on OX:
$\angle XOP = \theta$, where θ may be in degree or radians.
Now OMP is a right angled triangle,
We can form the six ratios as follows:

$\frac{a}{c}, \frac{b}{c}, \frac{a}{b}, \frac{b}{a}, \frac{c}{b}, \frac{c}{a}$



In fact these ratios depend only on the size of the angle and not on the triangle formed. Therefore these ratios called **Trigonometic ratios or**