Chapter 4
Partial Fractions

4.1 Introduction: A fraction is a symbol indicating the division of integers. For example, $\frac{13}{9}, \frac{2}{3}$ are fractions and are called Common Fraction. The dividend (upper number) is called the numerator N(x) and the divisor (lower number) is called the denominator, D(x).

From the previous study of elementary algebra we have learnt how the sum of different fractions can be found by taking L.C.M. and then add all the fractions. For example

i) $\frac{1}{x-1} + \frac{2}{x+2} = \frac{3x}{(x-1)(x+2)}$

ii) $\frac{2}{x+1} + \frac{1}{(x+1)^2} + \frac{3}{x-2} = \frac{9x^2+5x-3}{(x+1)^2(x-2)}$

Here we study the reverse process, i.e., we split up a single fraction into a number of fractions whose denominators are the factors of denominator of that fraction. These fractions are called **Partial fractions.**

4.2 Partial fractions :

To express a single rational fraction into the sum of two or more single rational fractions is called **Partial fraction resolution**.

For example,

$$\frac{2x + x^2 - 1}{x(x^2-1)} = \frac{1}{x} + \frac{1}{x-1} - \frac{1}{x+1}$$

$\frac{2x + x^2 - 1}{x(x^2-1)}$ is the resultant fraction and $\frac{1}{x} + \frac{1}{x-1} - \frac{1}{x+1}$ are its partial fractions.

4.3 Polynomial:

Any expression of the form $P(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$ where $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ are real constants, if $a_n \neq 0$ then P(x) is called polynomial of degree n.

4.4 Rational fraction:

We know that $\frac{p}{q}$, $q \neq 0$ is called a rational number. Similarly the quotient of two polynomials $\frac{N(x)}{D(x)}$ where $D(x) \neq 0$, with no common factors, is called a rational fraction. A rational fraction is of two types:

4.5 Proper Fraction:

A rational fraction $\frac{N(x)}{D(x)}$ is called a proper fraction if the degree of numerator N(x) is less than the degree of Denominator D(x).

For example

(i) $\frac{9x^2 - 9x + 6}{(x-1)(2x-1)(x+2)}$

(ii) $\frac{6x + 27}{3x^3 - 9x}$

4.6 Improper Fraction:

A rational fraction $\frac{N(x)}{D(x)}$ is called an improper fraction if the degree of the Numerator N(x) is greater than or equal to the degree of the Denominator D(x)

For example

(i) $\frac{2x^3 - 5x^2 - 3x - 10}{x^2 - 1}$

(ii) $\frac{6x^3 - 5x^2 - 7}{3x^2 - 2x - 1}$

Note: An improper fraction can be expressed, by division, as the sum of a polynomial and a proper fraction.

For example:

$$\frac{6x^3 + 5x^2 - 7}{3x^2 - 2x - 1} = (2x + 3) + \frac{8x - 4}{x^2 - 2x - 1}$$

Which is obtained as, divide $6x^2 + 5x^2 - 7$ by $3x^2 - 2x - 1$ then we get a polynomial (2x+3) and a proper fraction $\frac{8x - 4}{x^2 - 2x - 1}$

4.7 Process of Finding Partial Fraction:

A proper fraction $\frac{N(x)}{D(x)}$ can be resolved into partial fractions as:

(I) If in the denominator D(x) a linear factor (ax + b) occurs and is non-repeating, its partial fraction will be of the form $\frac{A}{ax + b}$,where A is a constant whose value is to be determined.

(II) If in the denominator D(x) a linear factor (ax + b) occurs n times, i.e., $(ax + b)^n$, then there will be n partial fractions of the form

$$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\frac{A_3}{(ax+b)^3}+\ldots\ldots+\frac{A_n}{(ax+b)^n}$$

,where A_1, A_2, A_3 - - - - - - - - - A_n are constants whose values are to be determined

(III) If in the denominator D(x) a quadratic factor $ax^2 + bx + c$ occurs and is non-repeating, its partial fraction will be of the form

$\frac{Ax+B}{ax^2+bx+c}$, where A and B are constants whose values are to be determined.

(IV) If in the denominator a quadratic factor $ax^2 + bx + c$ occurs n times, i.e., $(ax^2 + bx + c)^n$,then there will be n partial fractions of the form

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\frac{A_3x+B_3}{(ax^2+bx+c)^3}+$$

$$\text{- - - - - - - - - -}+\frac{A_nx+B_n}{(ax^2+bx+c)^n}$$

Where A_1, A_2, A_3 - - - - - - - - A_n and B_1, B_2, B_3 - - - - - - - B_n are constants whose values are to be determined.

Note: The evaluation of the coefficients of the partial fractions is based on the following theorem:

If two polynomials are equal for all values of the variables, then the coefficients having same degree on both sides are equal, for example , if

$px^2 + qx + a = 2x^2 - 3x + 5 \quad \forall x$, then

$p = 2, \quad q = -3$ and $a = 5$.

4.8 Type I

When the factors of the denominator are all linear and distinct i.e., non repeating.

Example 1:

Resolve $\frac{7x-25}{(x-3)(x-4)}$ into partial fractions.

Solution:

$$\frac{7x-25}{(x-3)(x-4)}=\frac{A}{x-3}+\frac{B}{x-4} \text{ ------------------(1)}$$

Multiplying both sides by L.C.M. i.e., $(x - 3)(x - 4)$, we get

$7x - 25 = A(x - 4) + B(x - 3)$ ------------- (2)

$7x - 25 = Ax - 4A + Bx - 3B$

$$7x - 25 = Ax + Bx - 4A - 3B$$
$$7x - 25 = (A + B)x - 4A - 3B$$

Comparing the co-efficients of like powers of x on both sides, we have

$$7 = A + B \text{ and}$$
$$-25 = -4A - 3B$$

Solving these equation we get

$$A = 4 \quad \text{and} \quad B = 3$$

Hence the required partial fractions are:

$$\frac{7x - 25}{(x - 3)(x - 4)} = \frac{4}{x-3} + \frac{3}{x-4}$$

Alternative Method:

Since $7x - 25 = A(x - 4) + B(x - 3)$

Put $x - 4 = 0, \Rightarrow x = 4$ in equation (2)

$$7(4) - 25 = A(4 - 4) + B(4 - 3)$$
$$28 - 25 = 0 + B(1)$$
$$B = 3$$

Put $x - 3 = 0 \Rightarrow x = 3$ in equation (2)

$$7(3) - 25 = A(3 - 4) + B(3 - 3)$$
$$21 - 25 = A(-1) + 0$$
$$-4 = -A$$
$$A = 4$$

Hence the required partial fractions are

$$\frac{7x - 25}{(x - 3)(x - 4)} = \frac{4}{x-3} + \frac{3}{x-4}$$

Note : The R.H.S of equation (1) is the identity equation of L.H.S

Example 2:

write the identity equation of $\frac{7x - 25}{(x-3)(x-4)}$

Solution : The identity equation of $\frac{7x - 25}{(x-3)(x-4)}$ is

$$\frac{7x - 25}{(x-3)(x-4)} = \frac{A}{x-3} + \frac{B}{x-4}$$

Example 3:

Resolve into partial fraction: $\frac{1}{x^2 - 1}$

Solutios: $\dfrac{1}{x^2 - 1} = \dfrac{A}{x - 1} + \dfrac{B}{x + 1}$

$1 = A(x + 1) + B(x - 1) \qquad (1)$

Put $x - 1 = 0, \Rightarrow x = 1$ in equation (1)

$1 = A(1 + 1) + B(1 - 1) \Rightarrow A = \dfrac{1}{2}$

Put $x + 1 = 0, \Rightarrow x = -1$ in equation (1)

$1 = A(-1+1) + B(-1-1)$

$1 = -2B, \Rightarrow B = \dfrac{1}{2}$

$$\frac{1}{x^2 - 1} = \frac{1}{2(x - 1)} - \frac{1}{2(x + 1)}$$

Example 4:

Resolve into partial fractions $\dfrac{6x^3 + 5x^2 - 7}{3x^2 - 2x - 1}$

Solution:

This is an improper fraction first we convert it into a polynomial and a proper fraction by division.

$$\frac{6x^3 + 5x^2 - 7}{3x^2 - 2x - 1} = (2x + 3) + \frac{8x - 4}{x^2 - 2x - 1}$$

Let $\dfrac{8x - 4}{x^2 - 2x - 1} = \dfrac{8x - 4}{(3x + 1)} = \dfrac{A}{x - 1} + \dfrac{B}{3x + 1}$

Multiplying both sides by $(x - 1)(3x + 1)$ we get

$$8x - 4 = A(3x + 1) + B(x - 1) \qquad (I)$$

Put $x - 1 = 0, \Rightarrow x = 1$ in (I), we get

The value of A

$$8(1) - 4 = A(3(1) + 1) + B(1 - 1)$$

$$8 - 4 = A(3 + 1) + 0$$

$$4 = 4A$$

$$\Rightarrow \quad A = 1$$

Put $3x + 1 = 0 \Rightarrow x = -\dfrac{1}{3}$ in (I)

$$8\left(-\frac{1}{3}\right)-4 = B\left(-\frac{1}{3}-1\right)$$

$$-\frac{8}{3}-4 = \left(-\frac{4}{3}\right)$$

$$-\frac{20}{3} = -\frac{4}{3}\ B$$

$$\Rightarrow \quad B = \frac{20}{3} x \frac{3}{4} = 5$$

Hence the required partial fractions are

$$\frac{6x^3 + 5x^2 - 7}{3x^2 - 2x - 1} = (2x + 3) + \frac{1}{x-1} + \frac{5}{3x+1}$$

Example 5:

Resolve into partial fraction $\dfrac{8x - 8}{x^3 - 2x^2 - 8x}$

Solution: $\dfrac{8x - 8}{x^3 - 2x^2 - 8x} = \dfrac{8x - 8}{x(x^2 - 2x - 8)} = \dfrac{8x - 8}{x(x-4)(x+2)}$

Let $\dfrac{8x - 8}{x^3 - 2x^2 - 8x} = \dfrac{A}{x} + \dfrac{B}{x-4} + \dfrac{C}{x+2}$

Multiplying both sides by L.C.M. i.e., $x(x – 4)(x + 2)$

$$8x – 8 = A(x – 4)(x + 2) + Bx(x + 2) + Cx(x – 4)$$

(I)

Put $x = 0$ in equation (I), we have

$$8 (0) – 8 = A(0 – 4)(0 + 2) + B(0)(0 + 2)+C(0)(0 – 4)$$

$$–8 = –8A + 0 + 0$$

$$\Rightarrow \quad A = 1$$

Put $x – 4 = 0 \quad \Rightarrow \quad x = 4$ in Equation (I), we have

$$8 (4) – 8 = B (4) (4 + 2)$$

$$32 – 8 = 24B$$

$$24 = 24B$$

$$\Rightarrow \quad B = 1$$

Put $x + 2 = 0 \Rightarrow x = – 2$ in Eq. (I), we have

$$8(–2) –8 = C(–2)(–2 –4)$$

$$–16 –8 = C(–2)(–6)$$

$$–24 = 12C$$

$$\Rightarrow \quad C = –2$$

Hence the required partial fractions

$$\frac{8x-8}{x^3-2x^2-8x}=\frac{1}{x}-\frac{1}{x-4}-\frac{2}{x+2}$$

Exercise 4.1

Resolve into partial fraction:

Q.1 $\dfrac{2x+3}{(x-2)(x+5)}$ Q.2 $\dfrac{2x+5}{x^2+5x+6}$

Q.3 $\dfrac{3x^2-2x-5}{(x-2)(x+2)(x+3)}$ Q.4 $\dfrac{(x-1)(x-2)(x-3)}{(x-4)(x-5)(x-6)}$

Q.5 $\dfrac{x}{(x-a)(x-b)(x-c)}$ Q.6 $\dfrac{1}{(1-ax)(1-bx)(1-cx)}$

Q.7 $\dfrac{2x^3-x^2+1}{(x+3)(x-1)(x+5)}$ Q.8 $\dfrac{1}{(1-x)(1-2x)(1-3x)}$

Q.9 $\dfrac{6x+27}{4x^3-9x}$ Q.10 $\dfrac{9x^2-9x+6}{(x-1)(2x-1)(x+2)}$

Q.11 $\dfrac{x^4}{(x-1)(x-2)(x-3)}$ Q.12 $\dfrac{2x^3+x^2-x-3}{x(x-1)(2x+3)}$

Answers 4.1

Q.1 $\dfrac{1}{x-2}+\dfrac{1}{x+5}$ Q.2 $\dfrac{1}{x+2}+\dfrac{1}{x+3}$

Q.3 $\dfrac{3}{20(x-2)}-\dfrac{11}{4(x-2)}+\dfrac{28}{5(x+3)}$

Q.4 $1+\dfrac{3}{x-4}-\dfrac{24}{x-5}+\dfrac{30}{x-6}$

Q.5 $\dfrac{a}{(a-b)(a-c)(x-a)}+\dfrac{b}{(b-a)(b-c)(x-b)}+\dfrac{c}{(c-b)(c-a)(x-c)}$

Q.6 $\dfrac{a^2}{(a-b)(a-c)(1-ax)}+\dfrac{b^2}{(b-a)(b-c)(1-bx)}+\dfrac{c^2}{(c-b)(c-a)(1-cx)}$

Q.7 $2+\frac{31}{4(x+3)}+\frac{1}{12(x-1)}-\frac{137}{6(x+5)}$

Q.8 $\frac{1}{2(1-x)}-\frac{4}{(1-2x)}+\frac{9}{2(1-3x)}$

Q.9 $\frac{3}{x}+\frac{4}{2x-3}+\frac{2}{2x+3}$

Q.10 $\frac{2}{x-1}-\frac{3}{2x-1}+\frac{4}{x+12}$

Q.11 $x+6+\frac{1}{2(x-1)}-\frac{16}{x-2}+\frac{81}{2(x-3)}$

Q.12 $1+\frac{1}{x}-\frac{1}{5(x-1)}-\frac{8}{5(2x+3)}$

4.9 Type II:

When the factors of the denominator are all linear but some are repeated.

Example 1:

Resolve into partial fractions: $\frac{x^2-3x+1}{(x-1)^2(x-2)}$

Solution:

$$\frac{x^2-3x+1}{(x-1)^2(x-2)}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x-2}$$

Multiplying both sides by L.C.M. i.e., $(x-1)^2(x-2)$, we get

$$x^2-3x+1 = A(x-1)(x-2)+B(x-2)+C(x-1)^2 \quad (I)$$

Putting $x-1=0 \quad \Rightarrow \quad x=1$ in (I), then

$(1)^2-3(1)+1 = B(1-2)$

$1-3+1 = -B$

$-1 = -B$

$\Rightarrow \quad B = 1$

Putting $x-2=0 \quad \Rightarrow \quad x=2$ in (I), then

$(2)^2-3(2)+1 = C(2-1)^2$

$4-6+1 = C(1)^2$

$\Rightarrow \quad -1 = C$

Now $x^2-3x+1 = A(x^2-3x+2)+B(x-2)+C(x^2-2x+1)$

Comparing the co-efficient of like powers of x on both sides, we get

$A + C = 1$

$A = 1 - C$