(iv) $32x^5 - 80x^4y + 80x^3y^2 - 40x^2y^3 + 10xy^4 - y^5$

(v) $128a^7 - 448a^5x + 672a^3x^2 - 560ax^3 + 280\frac{x^4}{a} - 84\frac{x^5}{a^3} + 14\frac{x^6}{a^5} - \frac{x^7}{a^7}$

(vi) $\frac{x^8}{y^8} - 8\frac{x^6}{y^6} + 28\frac{x^2}{y^2} - 56\frac{x^2}{y^2} + 70 - 56\frac{y^2}{x^2} + 28\frac{y^4}{x^4} - 8\frac{y^6}{x^6} + \frac{y^8}{x^8}$

(vii) $x^4 - 4x^3y^{-1} + 6x^2y^{-2} - 4xy^{-3} + y^{-4}$

2. (i) 1.14 (ii) 0.88 (iii) 34.47
3. (i) $2x^5 + 20x^3y^2 + 10xy^4$ (ii) $2x^4 + 24x^2 + 8$
4. (i) $1 - 4x + 10x^2 - 16x^3 + 19x^4 - 16x^5 + 10x^6 - 4x^7 + x^8$
   (ii) $16 + 32x - 8x^2 - 40x^3 + x^4 + 20x^5 - 2x^6 - 4x^7 + x^8$
5. (i) $1088640x^8$ (ii) $\frac{3003}{32}x^{20}y^5$ (iii) $\frac{101376}{x}$ (iv) $\frac{10500}{x^3}$
6. (i) $1913.625x^5$ (ii) $-\frac{77a^6b^5}{2592} + \frac{77a^5b^6}{3888}$ (iii) $\frac{280}{x} + 560x$
7. (i) $-1959552x^5$ (ii) $-252x^5$ (iii) $35x^9$ (iv) $-112x^2$
   (v) $\frac{880}{9}p^{16}q^8$
8. (i) -1959552 (ii) 46590 (iii) 33.185 (iv) $\frac{15}{2}a^{14}$
9. (i) 84 (ii) 5
10. (i) 7920 (ii) 672

3.7 Binomial Series

Since by the Binomial formula for positive integer n, we have

$$(a + b)^n = a^n + \frac{n}{1!}a^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \ldots\ldots\ldots + b^n \quad \ldots\ldots\ldots (2)$$

put a = 1 and b = x, then the above form becomes:

$$(1 + x)^n = 1 + \frac{n}{1!}x + \frac{n(n-1)}{2!}x^2 + \ldots\ldots + x^n$$

if n is –ve integer or a fractional number (-ve or +ve), then

$$(1+x)^n = 1 + \frac{n}{1!}x + \frac{n(n-1)}{2!}x^2 + \ldots\ldots\ldots\ldots\infty \qquad (3).$$

The series on the R.H.S of equation (3) is called binomial series.
This series is valid only when x is numerically less than unity i.e.,$|x| < 1$otherwise the expression will not be valid.

Note: The first term in the expression must be unity. For example, when n is not a positive integer (negative or fraction) to expand $(a + x)^n$, we shall have to write it as, $(a+x)^n = a^n\left(1+\frac{x}{a}\right)^n$ and then apply the binomial series, where $\left|\frac{x}{a}\right|$ must be less than 1.

3.8 Application of the Binomial Series; Approximations:

The binomial series can be used to find expression approximately equal to the given expressions under given conditions.

Example 1: If x is very small, so that its square and higher powers can be neglected then prove that

$$\frac{1+x}{1-x} = 1 + 2x$$

Solution:

$\frac{1+x}{1-x}$ this can be written as $(1 + x)(1 - x)^{-1}$

$= (1 + x)(1 + x + x^2 + \ldots\ldots\ldots$ higher powers of x)

$= 1 + x + x +$ neglecting higher powers of x.

$= 1 + 2x$

Example 2: Find to four places of decimal the value of $(1.02)^8$

Solution:

$$(1.02)^8 = (1 + 0.02)^8$$
$$= (1 + 0.02)^8$$
$$= 1 + \frac{8}{1}(0.02) + \frac{8.7}{2.1}(0.02)^2 + \frac{8.7.6}{3.2.1}(0.02)^3 + \ldots$$
$$= 1 + 0.16 + 0.0112 + 0.000448 + \ldots$$
$$= 1.1716$$

Example 3: Write and simplify the first four terms in the expansion of $(1 - 2x)^{-1}$.

Solution:

$$(1 - 2x)^{-1}$$
$$= [1 + (-2x)]^{-1}$$

Using $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + - - - - -$

$$= 1 + (-1)(-2x) + \frac{(-1)(-1-1)}{2!}(-2x)^2 + - - - - -$$

$$\frac{(-1)(-1-1)(-1-2)}{3!}(-2x)^3 + - - - - -$$

$$= 1 + 2x + \frac{(-1)(-2)}{2.1}4x^2 + \frac{(-1)(-2)(-3)}{3.2.1}(-8x^3) + - - - - -$$

$$= 1 + 2x + 4x^2 + 8x^3 + - - - -$$

Example 4: Write the first three terms in the expansion of $(2 + x)^{-3}$

Solution :

$$(2+x)^{-3} = (2)^{-3}\left(1 + \frac{x}{2}\right)^{-3}$$

$$= (2)^{-3}\left[1 + (-3)\left(\frac{x}{2}\right) + \frac{(-3)(-3-1)}{2!}\left(\frac{x}{2}\right)^2 + \cdots\right]$$

$$= \frac{1}{8}\left[1 - \frac{3}{2}x + 3x^2 + - - -\right]$$

Root Extraction:

The second application of the binomial series is that of finding the root of any quantity.

Example 5: Find square root of 24 correct to 5 places of decimals.
Solution:

$$\sqrt{24} = (25-1)^{1/2}$$

$$= (25)^{1/2}\left(1-\frac{1}{25}\right)^{1/2}$$

$$= 5\left(1-\frac{1}{5^2}\right)^{1/2}$$

$$= 5\left[1+\frac{1}{2}\left(-\frac{1}{5^2}\right)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(-\frac{1}{5^2}\right)^2 + - -\right]$$

$$= 5\left[1-\frac{1}{2.5^2}-\frac{1}{2^3.5^4}-\frac{1}{2^4.5^6}- ----\right]$$

$$= 5\,[1 - (0.02 + 0.0002 + 0.000004 + ---)]$$

$= 4.89898$

Example 6: evaluate $\sqrt[3]{29}$ to the nearest hundredth.

Solution :

$$\sqrt[3]{29} = (27+2)^{1/3} = \left[27\left(1+\frac{2}{27}\right)\right]^{1/3} = 3\left[1+\frac{2}{27}\right]^{1/3} + \ldots\ldots$$

$$= 3\left[1 + \frac{1}{3}\left(\frac{2}{27}\right) + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{1.2}\left(\frac{2}{27}\right)^2 + \ldots\ldots\ldots\right]$$

$$= 3\left[1 + \frac{2}{81} + \frac{1}{2}\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(\frac{2}{27}\right)^2 + \ldots\ldots\ldots\right]$$

$$= 3\,[1 + 0.0247 - 0.0006\ldots\ldots\ldots\ldots\ldots]$$

$$= 3\,[1.0212] = 3.07$$

Exercise 3.2

Q1: Expand upto four terms.

(i) $(1-3x)^{1/3}$ (ii) $(1-2x)^{-3/4}$ (iii) $(1+x)^{-3}$

(iv) $\dfrac{1}{\sqrt{1+x}}$ (v) $(4+x)^{1/2}$ (vi) $(2+x)^{-3}$

Q2: Using the binomial expansion, calculate to the nearest hundredth.

(i) $\sqrt[4]{65}$ (ii) $\sqrt{17}$ (iii) $(1.01)^{-7}$

(iv) $\sqrt{28}$ (v) $\sqrt{40}$ (vi) $\sqrt{80}$

Q3: Find the coefficient of x^5 in the expansion of

(i) $\dfrac{(1+x)^2}{(1-x)^2}$ (ii) $\dfrac{(1+x)^2}{(1-x)^3}$

Q4: If x is nearly equal to unity, prove that

$$\frac{mx^n - n\,x^m}{x^n - x^m} = \frac{1}{1-x}$$

Answers 3.2

Q1: (i) $1 - x - x - \frac{5}{3}x^3 + \text{----}$ (ii) $1 + \frac{3}{2}x + \frac{21}{8}x^2 + \frac{77}{16}x^3 + \text{----}$

(iii) $1 - 3x + 6x^2 - 10^3 \text{----}$ (iv) $1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \text{----}$

(v) $2+\frac{x}{2}-\frac{x^2}{64}+\frac{x^3}{512}+----$ (vi) $\frac{1}{8}\left[1-\frac{3}{2}x+\frac{3}{2}x^2-\frac{5}{4}x^3\right]$

Q2: (i) 2.84 (ii) 4.12 (iii) 0.93
(iv) 5.29 (v) 6.32 (vi) 8.94

Q3: (i) 20 (ii) 61

Summary

Binomial Theorem

An expression consisting of two terms only is called a binomial expression. If n is a positive index, then

1. The general term in the binomial expansion is $T_{r-1} = {}^nC_r a^{n-r} b^r$
2. The number of terms in the expansion of $(a + b)^n$ is n + 1.
3. The sum of the binomial coefficients in the expansion of $(a + b)^n$ is 2^n. i.e. ${}^nC_0 + {}^nC_1 + {}^nC_2 + ... + {}^nC_n = 2^n$
4. The sum of the even terms in the expansion of $(a + b)^n$ is equal to the sum of odd terms.
5. When n is even, then the only middle term is the $\left(\frac{n+2}{2}\right)$th term.
6. When n is odd, then there are two middle terms viz $\left(\frac{n+1}{2}\right)$th and $\left(\frac{n+3}{2}\right)$th terms.

Note: If n is not a positive index.

i.e. $(a + b)^n = a^n\left(1+\frac{n}{a}\right)^n$

$$= a^n\left[1+n\left(\frac{b}{a}\right)+\frac{n(n-1)}{2!}\left(\frac{b}{a}\right)^2+--------\right]$$

1. Here n is a negative or a fraction, the quantities ${}^nC_1$, ${}^nC_2$ ------- here no meaning at all. Hence co-efficients can not be represented as ${}^nC_1$, ${}^nC_2$ ---------.
2. The number of terms in the expansion is infinite as n is a negative or fraction.