Chapter 3
Binomial Theorem

3.1 Introduction:

An algebraic expression containing two terms is called a binomial expression, Bi means two and nom means term. Thus the general type of a binomial is $a + b$, $x - 2$, $3x + 4$ etc. The expression of a binomial raised to a small positive power can be solved by ordinary multiplication , but for large power the actual multiplication is laborious and for fractional power actual multiplication is not possible. By means of binomial theorem, this work reduced to a shorter form. This theorem was first established by Sir Isaac Newton.

3.2 Factorial of a Positive Integer:

If n is a positive integer, then the factorial of '*n*' denoted by *n*! or $\underline{|n}$ and is defined as the product of n +ve integers from n to 1 (or 1 to n)

i.e., $n! = n(n-1)(n-2) \ldots..$ 3.2.1

For example,

$$4! = 4.3.2.1 = 24$$

and $6! = 6.5.4.3.2.1 = 720$

one important relationship concerning factorials is that

$$(n+1)! = (n+1)\,n! \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_} \quad (1)$$

for instance,

$$5! = 5.4.3.2.1$$
$$= 5(4.3.2.1)$$
$$5! = 5.4!$$

Obviously, $1! = 1$ and this permits to define from equation (1)

$$n! = \frac{(n+1)!}{n+1}$$

Substitute 0 for n, we obtain

$$0! = \frac{(0+1)!}{0+1} = \frac{1!}{1} = \frac{1}{1}$$

$$0! = 1$$

3.3 Combination:

Each of the groups or selections which can be made out of a given number of things by taking some or all of them at a time is called combination.

In combination the order in which things occur is not considered e.g.; combination of a, b, c taken two at a time are ab, bc, ca.

The numbers $\binom{n}{r}$ **or** ${}^{n}C_{r}$

The numbers of the combination of n different objects taken 'r' at a time is denoted by $\binom{n}{r}$ **or** ${}^{n}C_{r}$ and is defined as,

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

e.g,

$$\binom{6}{4} = \frac{6!}{4!(6-4)!}$$

$$= \frac{6\times5\times4!}{4!\times 2!} = \frac{6\times5}{2\times1} = 15$$

Example 1: Expand $\binom{7}{3}$

Solution. $\binom{7}{3} = \frac{7!}{3!(7-3)!}$

$$= \frac{7.6.5.4!}{3.2.1.4!}$$

$= 35$

This can also be expand as

$$\binom{7}{3} = \frac{7.6.5}{3.2.1} = 35$$

If we want to expand $\binom{7}{5}$, then

$$\binom{7}{5} = \frac{7.6.5.4.3}{5.4.3.2.1} = 21$$

Procedure: Expand the above number as the lower number and the lower number expand till 1.

Method 2

For expansion of $\binom{n}{r}$ we can apply the method:

a. If r is less than (n – r) then take r factors in the numerator from n to downward and r factors in the denominator ending to 1.

b. If n – r is less than r, then take (n – r) factors in the numerator from n to downward and take (n – r) factors in the denominator ending to 1. For example, to expand $\binom{7}{5}$ again, here 7 – 5 = 2 is less than 5, so take two factors in numerator and two in the denominator as,

$$\binom{7}{5} = \frac{7.6}{2.1} = 21$$

Some Important Results

(i). $\binom{n}{0} = \frac{n!}{0!(n-0)!} = \frac{n!}{1 \times n!} = 1$

(ii) $\binom{n}{n} = \frac{n!}{n!(n-n)!} = \frac{n!}{n! \times 0!} = \frac{n!}{n! \times 1} = 1$

(iii) $\binom{n}{r} = \binom{n}{n-r}$

For example

$\binom{4}{0} = \binom{4}{4} = 1$ as $\frac{4!}{0!(4-0)!} = \frac{4!}{4!\,0!}$

$1 = 1$

$\binom{4}{3} = \binom{4}{1} = 4$ as $\frac{4!}{3!.1!} = \frac{4!}{1!.\ 3!}$

$$\frac{4.\ 3!}{3!.1!} = \frac{4.\ 3!}{1!.\ 3!}$$

$$4 = 4$$

Note: The numbers $\binom{n}{r}$ **or** ${}^{n}C_{r}$ are also called binomial co-efficients

3.4 The Binomial Theorem:

The rule or formula for expansion of $(a + b)^n$, where n is any positive integral power , is called binomial theorem .

For any positive integral n

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2} + \binom{n}{3}a^{n-3}b^3 \dots\dots$$

$$+\binom{n}{r}a^{n-r}b^r \dots\dots\dots.. + \binom{n}{n}b^n \text{-------------------(1)}$$

or briefly, $$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r$$

Remarks:- The coefficients of the successive terms are

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, -----, \binom{n}{r} -----, \binom{n}{n}$$

and are called **Binomial coefficients.**

Note : Sum of binomial coefficients is 2^n

Another form of the Binomial theorem:

$$(a+b)^n = a^n + \frac{n}{1!}a^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \ldots\ldots +$$

$$\frac{n(n-1)(n-2)------(n-r+1)}{r!}a^{n-r}b^r + \ldots\ldots\ldots + b^n ----------------- (2)$$

Note: Since,

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

So, $$\binom{n}{0} = \frac{n!}{0!(n-0)!} = \frac{n!}{1 \times n!} = 1$$

$$\binom{n}{1} = \frac{n!}{1!(n-1)!} = \frac{n(n-1)!}{1!(n-1)!} = \frac{n}{1!}$$

$$\binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{n(n-2)!}{2!(n-2)!} = \frac{n(n-1)}{2!}$$

$$\binom{n}{3} = \frac{n!}{3!(n-3)!} = \frac{n(n-1)(n-2)(n-3)!}{3!(n-3)!} = \frac{n(n-1)(n-2)}{3!}$$

$$\binom{n}{r} = \frac{n(n-1)(n-2)!\ldots\ldots(n-r+1)(n-r)!}{r!(n-r)!}$$

$$= \frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{r!}$$

$$\binom{n}{n} = \frac{n!}{n!(n-n)!} = \frac{n!}{n! \times 0!} = \frac{n!}{n! \times 1} = 1$$

The following points can be observed in the expansion of $(a + b)^n$

1. There are (n + 1) terms in the expansion.
2. The 1^{st} term is a^n and (n + 1)th term or the last term is b^n
3. The exponent of 'a' decreases from n to zero.
4. The exponent of 'b' increases from zero to n.
5. The sum of the exponents of a and b in any term is equal to index n.
6. The co-efficients of the term equidistant from the beginning and end of the expansion are equal as $\binom{n}{r} = \binom{n}{n-r}$

3.5 General Term:

The term $\binom{n}{r} a^{n-r} b^r$ in the expansion of binomial theorem is called the General term or (r + 1)th term. It is denoted by T_{r+1}. Hence

$$T_{r+1} = \binom{n}{r} a^{n-r} b^r$$

Note: The General term is used to find out the specified term or the required co-efficient of the term in the binomial expansion

Example 2: Expand $(x + y)^4$ by binomial theorem:

Solution:

$$(x + y)^4 = x^4 + \binom{4}{1} x^{4-1} y + \binom{4}{2} x^{4-2} y^2 + \binom{4}{3} x^{4-3} y^3 + y^4$$

$$= x^4 + 4x^3y + \frac{4 \times 3}{2 \times 1} x^2y^2 + \frac{4 \times 3 \times 2}{3 \times 2 \times 1} xy^3 + y^4$$

$$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

Example 3: Expand by binomial theorem $\left(a - \frac{1}{a}\right)^6$

Solution:

$$\left(a - \frac{1}{a}\right)^6 = a^6 + \binom{6}{1} a^{6-1}\left(-\frac{1}{a}\right)^1 + \binom{6}{2} a^{6-2}\left(-\frac{1}{a}\right)^2 + \binom{6}{3} a^{6-3}\left(-\frac{1}{a}\right)^3 +$$

$$\binom{6}{4} a^{6-4}\left(-\frac{1}{a}\right)^4 + \binom{6}{5} a^{6-5}\left(-\frac{1}{a}\right)^5 + \binom{6}{6} a^{6-6}\left(-\frac{1}{a}\right)^6$$

$$= a^6 + 6a^5\left(-\frac{1}{a}\right) + \frac{6 \times 5}{2 \times 1} a^4\left(-\frac{1}{a^2}\right) + \frac{6 \times 5 \times 4}{3 \times 2 \times 1} a^3\left(-\frac{1}{a^3}\right) +$$

$$\frac{6 \times 5 \times 4 \times 3}{4 \times 3 \times 2 \times 1} a^2\left(-\frac{1}{a^4}\right)+\frac{6 \times 5 \times 4 \times 3 \times 2}{5 \times 4 \times 3 \times 2 \times 1} a\left(-\frac{1}{a^5}\right)^5+\left(-\frac{1}{a^6}\right)$$

$$= a^6 - 6a^4 + 15a^2 - 20 + \frac{15}{a^2} - \frac{6}{a^5} + \frac{1}{a^6}$$

Example 4: Expand $\left(\frac{x^2}{2} - \frac{2}{x}\right)^4$

Solution:

$$\left(\frac{x^2}{2} - \frac{2}{x}\right)^4 = \left(\frac{x^2}{2}\right)^4 + \binom{4}{1}\left(\frac{x^2}{2}\right)^{4-1}\left(\frac{-2}{x}\right)^1 + \binom{4}{2}\left(\frac{x^2}{2}\right)^{4-2}\left(\frac{-2}{x}\right)^2$$

$$+ \binom{4}{3}\left(\frac{x^2}{2}\right)^{4-3}\left(\frac{-2}{x}\right)^3 + \binom{4}{4}\left(\frac{x^2}{2}\right)^{4-4}\left(\frac{-2}{x}\right)^4$$

$$= \frac{x^4}{16} + 4\left(\frac{x^2}{2}\right)^3\left(-\frac{2}{x}\right) + \frac{4\,.\,3}{2\,.\,1}\left(\frac{x^2}{2}\right)^2\left(\frac{4}{x^2}\right) +$$

$$\frac{4\,.\,3\,.\,2}{3\,.\,2\,.\,1}\left(\frac{x^2}{2}\right)\left(-\frac{8}{x^3}\right) + \frac{16}{x^4}$$

$$= \frac{x^8}{16} - 4.\frac{x^8}{8}.\frac{2}{x} + 6.\frac{x^4}{4}.\frac{4}{x^2} - 4\frac{x^2}{2}.\frac{8}{x^3} + \frac{16}{x^4}$$

$$= \frac{x^8}{16} - x^5 + 6x^2 - \frac{16}{x} + \frac{16}{x^4}$$

Example 5: Expand $(1.04)^5$ by the binomial formula and find its value to two decimal places.

Solution:

$$(1.04)^5 = (1 + 0.04)^5$$

$$(1 + 0.04)^5 = (1)^5 + \binom{5}{1}(1)^{5-1}(0.04) + \binom{5}{2}(1)^{5-2}(0.04)^2 + \binom{5}{3}$$

$$(1)^{5-3}(0.04)^3 + \binom{5}{4}(1)^{5-4}(0.04)^4 + (0.04)^5$$

$$= 1. + 0.2 + 0.016 + 0.00064 + 0.000128$$

$$+ 0.000\,000\,1024$$

$$= 1.22$$

Example 6: Find the eighth term in the expansion of $\left(2x^2 - \frac{1}{x^2}\right)^{12}$

Solution: $\left(2x^2 - \frac{1}{x^2}\right)^{12}$

The General term is, $T_{r+1} = \binom{n}{r} a^{n-r} b^r$

Here $T_8 = ?$ $a = 2x^2$, $b = -\frac{1}{x^2}$, $n = 12$, $r = 7$,

Therefore, $T_{7+1} = \binom{12}{7}\left(2x^2\right)^{12-7}\left(-\frac{1}{x^2}\right)^7$

$$T_8 = \frac{12 . 11 . 10 . 9 . 8 . 7 . 6}{7 . 6 . 5 . 4 . 3 . 2 . 1}\left(2x^2\right)^5 \frac{(-1)^7}{x^{14}}$$

$$T_8 = 793 \times 32x^{10} \frac{(-1)}{x^{14}}$$

$$T_8 = -\frac{25\,344}{x^4}$$

Eighth term $= T_8 = -\dfrac{25\,344}{x^4}$

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3.6 Middle Term in the Expansion $(a + b)^n$

In the expansion of $(a + b)^n$, there are $(n + 1)$ terms.

Case I :

If n is even then $(n + 1)$ will be odd, so $\left(\frac{n}{2} + 1\right)$th term will be the only one middle term in the expension .

For example, if $n = 8$ (even), number of terms will be 9 (odd), therefore, $\left(\frac{8}{2} + 1\right) = 5^{th}$ will be middle term.

Case II:

If n is odd then $(n + 1)$ will be even, in this case there will not be a single middle term, but $\left(\frac{n+1}{2}\right)$th and $\left(\frac{n+1}{2} + 1\right)$th term will be the two middle terms in the expension.

For example, for $n = 9$ (odd), number of terms is 10 i.e. $\left(\frac{9+1}{2}\right)$th and $\left(\frac{9+1}{2}+1\right)$th i.e. 5^{th} and 6^{th} terms are taken as middle terms and these middle terms are found by using the formula for the general term.

Example 7: Find the middle term of $\left(1-\frac{x^2}{2}\right)^{14}$.

Solution:

We have n = 14, then number of terms is 15.

$\therefore \left(\frac{14}{2}+1\right)$ i.e. 8^{th} will be middle term.

$a = 1,\ b = -\frac{x^2}{2}, \qquad n = 14, \quad r = 7, \quad T_8 = ?$

$$T_{r+1} = \binom{n}{r} a^{n-r} b^r$$

$$T_{7+1} = \binom{14}{7} (1)^{14-7} \left(-\frac{x^2}{2}\right)^7 = \frac{14!}{7!7!} (-1)^7 \frac{x^{14}}{2^7}$$

$$T_8 = \frac{14.13.12.11.10.9.8.7!}{7.6.5.4.3.2.1\ 7!} \frac{(-1)}{128}.x^{14}$$

$$T_8 = -(2)(13)(11)(2)(3) \frac{1}{128}.\ x^{14}$$

$$T_8 = -\frac{429}{16}\ x^{14}$$

Example 8 : Find the coefficient of x^{19} in $(2x^3 - 3x)^9$.

Solution:

Here, $a = 2x^3$, $\quad b = -3x$, $\quad n = 9$

First we find r.

$$\text{Since } T_{r+1} = \binom{n}{r} a^{n-r} b^r$$

$$= \binom{9}{r} (2x^3)^{9-r} (-3x)^r$$

$$= \binom{9}{r} 2^{9-r} (-3)^r x^{27-3r}.x^r$$

$$= \binom{9}{r} 2^{9-r} (-3)^r .x^{27-2r} \quad \ldots\ldots\ldots\ldots\ldots (1)$$

But we require x^{19}, so put

$19 = 27 - 2r$

$2r = 8$

$r = 4$

Putting the value of r in equation (1)

$$T_{4+1} = \binom{9}{r} 2^{9-4}(-3)^4 x^{19}$$

$$= \frac{9.8.7.6.5}{4.3.2.1} . 2^5 . 3^4 x^{19}$$

$$= 630 \text{ x } 32 \text{ x } 81 \; x^{19}$$

$$T_5 = 1632960 \; x^{19}$$

Hence the coefficient of x^{19} is 1632960

Example 9: Find the term independent of x in the expansion of $\left(2x^2 + \frac{1}{x}\right)^9$.

Solution:

Let T_{r+1} be the term independent of x.

We have $a = 2x^2$, $b = \frac{1}{x}$, $n = 9$

$$T_{r+1} = \binom{n}{r} a^{n-r} b^r = \binom{9}{r}\left(2x^2\right)^{9-r}\left(\frac{1}{x}\right)^r$$

$$T_{r+1} = \binom{9}{r} 2^{9-r} . x^{18-2r} . x^r$$

$$T_{r+1} = \binom{9}{r} 2^{9-r} . x^{18-3r} \text{ .... (1)}$$

Since T_{r+1} is the term independent of x i.e. x^0.

$\therefore$ power of x must be zero.

i.e. $18 - 3r = 0 \Rightarrow r = 6$

put in (1)

$$T_{r+1} = \binom{9}{6} 2^{9-6} . x^0 = \frac{!9}{!6!3^{2^3}} . 1$$

$$= \frac{{}^3\not{9} . \not{8}^4 . 7.6!}{6! . 3 . \not{2} . 1} . 8 . 1 = 672$$

Exercise 3.1

1. Expand the following by the binomial formula.

   (i) $\left(x+\frac{1}{x}\right)^4$ (ii) $\left(\frac{2x}{3}-\frac{3}{2x}\right)^5$ (iii) $\left(\frac{x}{2}-\frac{2}{y}\right)^4$

   (iv) $(2x-y)^5$ (v) $\left(2a-\frac{x}{a}\right)^7$ (vi) $\left(\frac{x}{y}-\frac{y}{x}\right)^4$

   (vii) $(-x+y^{-1})^4$

2. Compute to two decimal places of decimal by use of binomial formula.

   (i) $(1.02)^4$ (ii) $(0.98)^6$ (iii) $(2.03)^5$

3. Find the value of

   (i) $(x+y)^5+(x-y)^5$ (ii) $(x+\sqrt{2})^4+(x-\sqrt{2})^4$

4. Expanding the following in ascending powers of x

   (i) $(1-x+x^2)^4$ (ii) $(2+x-x^2)^4$

5. Find

   (i) the 5^{th} term in the expansion of $\left(2x^2-\frac{3}{x}\right)^{10}$

   (ii) the 6^{th} term in the expansion of $\left(x^2+\frac{y}{2}\right)^{15}$

   (iii) the 8^{th} term in the expansion of $\left(\sqrt{x}+\frac{2}{\sqrt{x}}\right)^{12}$

   (iv) the 7^{th} term in the expansion of $\left(\frac{4x}{5}-\frac{5}{2x}\right)^9$

6. Find the middle term of the following expansions

   (i) $\left(3x^2+\frac{1}{2x}\right)^{10}$ (ii) $\left(\frac{a}{2}-\frac{b}{3}\right)^{11}$ (iii) $\left(2x+\frac{1}{x}\right)^7$

7. Find the specified term in the expansion of

   (i) $\left(2x^2-\frac{3}{x}\right)^{10}$: term involving x^5

(ii) $\left(2x^2 - \frac{1}{2x}\right)^{10}$: term involving x^5

(iii) $\left(x^3 + \frac{1}{x}\right)^7$: term involving x^9

(iv) $\left(\frac{x}{2} - \frac{4}{x}\right)^8$: term involving x^2

(v) $\left(\frac{p^2}{2} + 6q^2\right)^{12}$: term involving q^8

8. Find the coefficient of

(i) x^5 in the expansion of $\left(2x^2 - \frac{3}{x}\right)^{10}$

(ii) x^{20} in the expansion of $\left(2x^2 + \frac{1}{2x}\right)^{16}$

(iii) x^5 in the expansion of $\left(2x^2 - \frac{1}{3x}\right)^{10}$

(iv) b^6 in the expansion of $\left(\frac{a^2}{2} + 2b^2\right)^{10}$

9. Find the constant term in the expansion of

(i) $\left(x^2 - \frac{1}{x}\right)^9$ (ii) $\left(\sqrt{x} + \frac{1}{3x^2}\right)^{10}$

10. Find the term independent of x in the expansion of the following

(i) $\left(2x^2 - \frac{1}{x}\right)^{12}$ (ii) $\left(2x^2 + \frac{1}{x}\right)^9$

Answers 3.1

1. (i) $x^4 + 4x^2 + 6 + \frac{4}{x^2} + \frac{1}{x^4}$

(ii) $\frac{32}{243}x^5 - \frac{40}{27}x^3 + \frac{20}{3}x - \frac{15}{x} + \frac{135}{8x^3} - \frac{243}{32x^5}$

(iii) $\frac{x^4}{16} - \frac{x^3}{y} + \frac{6x^2}{y^2} - \frac{6x}{y^3} + \frac{16}{y^4}$

(iv) $32x^5 - 80x^4y + 80x^3y^2 - 40x^2y^3 + 10xy^4 - y^5$

(v) $128a^7 - 448a^5x + 672a^3x^2 - 560ax^3 + 280\frac{x^4}{a} -$

$84\frac{x^5}{a^3} + 14\frac{x^6}{a^5} - \frac{x^7}{a^7}$

(vi) $\frac{x^8}{y^8} - 8\frac{x^6}{y^6} + 28\frac{x^2}{y^2} - 56\frac{x^2}{y^2} + 70 - 56\frac{y^2}{x^2} + 28\frac{y^4}{x^4} - 8\frac{y^6}{x^6} + \frac{y^8}{x^8}$

(vii) $x^4 - 4x^3y^{-1} + 6x^2y^{-2} - 4xy^{-3} + y^{-4}$

2. (i) 1.14 (ii) 0.88 (iii) 34.47
3. (i) $2x^5 + 20x^3y^2 + 10xy^4$ (ii) $2x^4 + 24x^2 + 8$
4. (i) $1 - 4x + 10x^2 - 16x^3 + 19x^4 - 16x^5 + 10x^6 - 4x^7 + x^8$
   (ii) $16 + 32x - 8x^2 - 40x^3 + x^4 + 20x^5 - 2x^6 - 4x^7 + x^8$
5. (i) $1088640x^8$ (ii) $\frac{3003}{32}x^{20}y^5$ (iii) $\frac{101376}{x}$ (iv) $\frac{10500}{x^3}$
6. (i) $1913.625x^5$ (ii) $-\frac{77a^6b^5}{2592} + \frac{77a^5b^6}{3888}$ (iii) $\frac{280}{x} + 560x$
7. (i) $-1959552x^5$ (ii) $-252x^5$ (iii) $35x^9$ (iv) $-112x^2$
   (v) $\frac{880}{9}p^{16}q^8$
8. (i) -1959552 (ii) 46590 (iii) 33.185 (iv) $\frac{15}{2}a^{14}$
9. (i) 84 (ii) 5
10. (i) 7920 (ii) 672

3.7 Binomial Series

Since by the Binomial formula for positive integer n, we have

$$(a + b)^n = a^n + \frac{n}{1!}a^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \ldots\ldots + b^n \qquad \ldots\ldots (2)$$

put a = 1 and b = x, then the above form becomes:

$$(1 + x)^n = 1 + \frac{n}{1!}x + \frac{n(n-1)}{2!}x^2 + \ldots\ldots + x^n$$

if n is –ve integer or a fractional number (-ve or +ve), then