2.13 Geometric Series

A geometric series is the sum of the terms of a geometric sequence.

If a, ar, ar^2, …… + ar^{n-1} is a geometric sequence.

Then $a + ar + ar^2 + \ldots\ldots + ar^{n-1}$ is a geometric series.

Sum of n Terms of a Geometric Series

Let, S_n be the sum of geometric series

i.e. $S_n = a + ar + ar^2 + \ldots\ldots + ar^{n-1}$ ……………….. (1)

Multiplying by r on both sides

$rS_n = ar + ar^2 + ar^3 + \ldots\ldots + ar^{n-1} + ar^n$……………….. (2)

Subtracting (2) from (1), we get

$S_n - r S_n = a - a r^n$

$(1 - r)S_n = a(1 - r^n)$

$$S_n = \frac{a(1-r^n)}{1-r} \quad ; \; r \neq 1$$

For convenience, we use :

$$S_n = \frac{a(1-r^n)}{1-r} \quad \text{if } |r| < 1$$

and $$S_n = \frac{a(r^n-1)}{r-1} \quad \text{if } |r| > 1$$

Example 1:

Sum the series $\frac{2}{3}, -1, \frac{3}{2}, \ldots\ldots\ldots$ to 7 terms

Solution

Here $a = \frac{2}{3}$, $r = \frac{-1}{\frac{2}{3}} = \frac{-3}{2}$

$$S_n = \frac{a(1-r^n)}{1-r} \text{ (because } r < 1)$$

$$S_7 = \frac{\frac{2}{3}\left[1-\left(-\frac{3}{2}\right)^7\right]}{1-\left(-\frac{3}{2}\right)} = \frac{\frac{2}{3}\left[1+\frac{2187}{128}\right]}{\frac{5}{2}}$$

$$S_7 = \frac{2}{3}\left(\frac{2315}{128}\right)\frac{2}{5} = \frac{463}{96}$$

Example 2:

Sum to 5 terms the series 1 + 3 + 9 + ……..

Solution:

The given series is a G.P.

in which a = 1, $r = \frac{a_2}{a_1} = \frac{3}{1} = 3,\ n = 5$

$\therefore \quad S_n = \frac{a(r^4 - 1)}{r - 1}$ (because r > 1)

$$S_n = \frac{1[(3)^5 - 1]}{3-1} = \frac{243-1}{2} = \frac{242}{2} = 121$$

Example 3:

Find S_n for the series 2 + 4 + 8 + ……….. + 2^n.

$\therefore$ Since r = r > 1

$$\therefore \quad S_n = \frac{a(r^n - 1)}{r-1} = \frac{2(2^n - 1)}{2-1} = 2^{n+1} - 2$$

Example 4:

How many terms of the series

$$\frac{2}{3} - \frac{1}{3} + \frac{1}{2} + \ \ldots\ldots\ldots \text{ amount to } \frac{55}{72}$$

Solution:

$$S_n = \frac{55}{72},\ n = ?\ \ a = \frac{2}{9},\ \ r = \frac{-\frac{1}{3}}{\frac{2}{9}} = \frac{-3}{2}$$

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

$$\frac{55}{72} = \frac{\frac{2}{9}\left[1 - \left(-\frac{3}{2}\right)^n\right]}{1 - \left(-\frac{3}{2}\right)} = \frac{\frac{2}{9}\left[1 - \left(-\frac{3}{2}\right)^n\right]}{\frac{3}{2}}$$

$$\frac{55}{72} = \frac{4}{45}\left[1 - \left(-\frac{3}{2}\right)^n\right]$$

$$\frac{45x55}{72x4} = 1 - \left(-\frac{3}{2}\right)^n \qquad \Rightarrow \frac{275}{32} = 1 - \left(-\frac{3}{2}\right)^n$$

$$\left(-\frac{3}{2}\right)^n = 1-\frac{275}{32} = \frac{243}{32} = \left(-\frac{3}{2}\right)^5$$

$\Rightarrow n = 5$

Example 5:

Sum the series:

(i) 0.2 + .22 + .222 + …… to n terms

(ii) $(x + y)(x^2 + xy + y^2) + (x^3 + x^2y + xy^2 + y^3) + \ldots\ldots$ to n terms.

Solutions:

(i) 0.2 + .22 + .222 + …… to n terms

Let, $S_n = .2 + .22 + .222 + \ldots\ldots$ to n terms

$= 2[.1 + .11 + .111 + \ldots\ldots$ to n terms]

Multiplying and dividing by 9

$$S_n = \frac{2}{9}[.9 + .99 + .999 + \ldots\ldots \text{ to n terms}]$$

$$= \frac{2}{9}[(1 - 1) + (1 - .01) + (0.1 - .001) + \ldots\ldots \text{ to n terms}]$$

$$= \frac{2}{9}\ [(1+1+1+\ldots\ldots \text{n terms})-(0.1+.01+.001+\ldots\ldots\text{to n terms})]$$

$a = .1 \qquad r = \dfrac{.01}{.1} = 0.1 = \dfrac{1}{10}$

$a = \dfrac{1}{10}$

We use $S_n = \dfrac{a(1-r^n)}{1-r}$

$$S_n = \frac{2}{9}\left[n - \frac{\frac{1}{10}\left\{1-\left(\frac{1}{10}\right)^n\right\}}{1-\frac{1}{10}}\right] = \frac{2}{9}\left[n - \frac{1}{9}\left\{1-\frac{1}{10^n}\right\}\right]$$

Solution (ii)

Let, $S_n = (x + y) + (x^2 + xy + y^2) + (x^3+x^2y+xy^2+y^3) + \ldots.$ to n term.

Multiplying and dividing by $(x - y)$

$$S_n = \frac{1}{(x-y)}\ [(x+y)(x-y)+(x-y)(x^2+xy+y^2)+(x-y)\ (x^3+x^2y+xy^2+y^3)+ \ldots$$

$$S_n = \frac{1}{(x-y)}\ [(x^2 - y^2)+ (x^3 - y^3)+ (x^4 - y^4)+ \ldots. \text{ to n term}]$$

$$S_n = \frac{1}{(x-y)}[(x^2 + x^3 + x^4 + \ldots \text{ to n term}) - (y^2 + x^3 + y^4 + \ldots \text{ to n term}]$$

We use $S_n = \dfrac{a(r^n - 1)}{r - 1}$

$$S_n = \frac{1}{(x-y)}\left[\frac{x^2(x^n - 1)}{x - 1} - \frac{y^2(y^n - 1)}{y - 1}\right]$$

Example 6:

The sum of the first 10 terms of a G.P. is equal to 244 times the sum of first 5 terms. Find common ratio.

Solution:

Here, $n = 10, n = 5, r = ?$

So, $S_n = \dfrac{a(1 - r^n)}{1 - r}$

$$S_{10} = \frac{a(1 - r^{10})}{1 - r}, \quad S_5 = \frac{a(1 - r^5)}{1 - r}$$

By the Given condition:

$S_{10} = 244S_5$

$$\frac{a(1 - r^{10})}{1 - r} = 244\left[\frac{a(1 - r^5)}{1 - r}\right]$$

$\Rightarrow 1 - r^{10} = 244(1 - r^5)$

$(1)^2 - (r^2)^5 = 244(1 - r^5)$

$(1 - r^5)(1 + r^5) = 244(1 - r^5) \quad \Rightarrow (1 - r^5)[1 + r^5 - 244] = 0$

$\Rightarrow 1 + r^5 - 244 = 0$ or $1 - r^5 = 0$

$1 + r^5 = 244 = 0 \qquad r^5 = 1$

$r^5 = 243 \quad \Rightarrow \boxed{r = 3} \qquad r = 1$ which not possible

Example 7:

Given $n = 6, r = \dfrac{2}{3}, S_n = \dfrac{665}{144}$ find a.

Solution:

Formula $S_n = \dfrac{a(r^n - 1)}{r - 1} \qquad \because |r| > 1$

$$\frac{665}{144} = \frac{a\left[1 - \left(\frac{2}{3}\right)^6\right]}{1 - \frac{2}{3}}$$

$$= \frac{a\left[1-\frac{64}{729}\right]}{\frac{1}{3}}$$

$$\frac{665}{144} = a\left[\frac{665}{243}\right]$$

$$a = \frac{665}{144} \times \frac{243}{665}$$

$$\boxed{a = \frac{27}{16}}$$

Example 8:

If a man deposits \$ 200 at the beginning of each year in a bank that pays 4 percent compounded annually, how much will be to his credit at the end of 6 years?

Solution:

The man deposits \$ 200 at the beginning of each year.
The bank pays 4% compounded interest annually
At the end of first year the principle amount or credit becomes
$= 200(1.04)$
At the beginning of second year the principle amount or credit is
$= 200 + 200\ (1.04)$
At the end of second year the principle amount or credit becomes

$$= 200(1.04) + 200(1.04)^2$$
$$= 200(1.04 + 1.04^2)$$

So at the end of 6 years the principle amount or credit becomes
$= 200\ (1.04 + 1.04^2 + \ldots$ sum upto 6 times)
Consider, $1.04 + 1.04^2 + \text{- - - - - - - -}$ 6 terms.

$a = 1.04,\ r = 1.04,$ and $n = 6$

By the formula

$$S_n = \frac{a(r^n - 1)}{r - 1} \qquad \because |r| > 1$$

$$S_6 = \frac{1.04(1.04^6 - 1)}{1.04 - 1}$$

$$= \frac{1.04(1.2653 - 1)}{0.04}$$

$$= \frac{1.04 \times 0.2653}{0.04}$$

$$= 6.8983$$

Hence at the end of 6 years the credit is $= 200(6.8983)$
$= \$1379.66$

Exercise 2.6

Q1. Find the sum of each of the following series:

(i) $1+\frac{1}{3}+\frac{1}{9}+$ ------- to 6 terms

(ii) $x + x^2 + x^3$ ------- to 20 terms.

(iii) $\frac{1}{8}+\frac{1}{4}+\frac{1}{2}+$ -------- $+ 64$

(iv) $3 + 3^2 + 3^3 +$ ---------- $+ 3^n$

Q2. How many terms of the series?

$\frac{2}{3}-1+\frac{3}{2}-\frac{9}{4}+$ ------- amount to $-\frac{133}{48}$

Q3. Sum the series.

(i) $.3 + .33 + .333 +$ ------- to n terms.

(ii) $3 + 33 + 333 +$ ------- to n terms.

(iii) $1 + (1 + x)r + (1+ x+x^2)r^2+(1+x+x^2+x^3)r^3 +$ - - - to n terms.

Q4. What is the sum of the geometric series for which $a = 2$, $n = 5$, $l = a_n = 32$?

Q5. A rubber ball is dropped from a height of 4.8 dm. It continuously rebounds, each time rebounding $\frac{3}{4}$ of the distance of the preceding fall. How much distance has it traveled when it strikes the ground for the sixth time?

Q6. The first term of geometric progression is $\frac{1}{2}$ and the 10th term is 256, using formula find sum of its 12 terms.

Q7. What is first term of a six term G.P. in which the common ratio is $\sqrt{3}$ and the sixth term is 27 find also the sum of the first three terms.

Answers 2.6

1. (i) $\frac{364}{243}$ (ii) $\frac{x(1-x^{20})}{1-x}$ (iii) 1023/8

(iv) $\frac{3(3^n - 1)}{2}$

2. n = 6

3. (i) $\frac{1}{3}\left[n - \frac{1}{9}\left(1 - \frac{1}{10^n}\right)\right]$ (ii) $\frac{1}{3}\left[\frac{10(10^n - 1)}{9} - n\right]$

(iii) $\frac{1}{(1-x)}\left[\frac{1-r^n}{1-r} - \frac{x(1-r^n x^n}{1-rx}\right]$

4. 62 5. 26.76 6. $\frac{4085}{2}$ 7. $\sqrt{3}; \frac{\sqrt{3}(3\sqrt{3}-1)}{\sqrt{3}-1}$

2.14 Infinite Geometric Sequence:

A geometric sequence in which the number of terms are infinite is called as infinite geometric sequence.

For example:

(i) $2, \frac{4}{3}, \frac{8}{9}, \frac{16}{27}$, - - - - - -

(ii) 2, 4, 8, 16, 32, - - - - - - -

Infinite Series:

Consider a geometric sequence a, ar, ar^2, - - - - - - - to n terms.

Let S_n denote the sum of n terms then $S_n = a + ar + ar^2 + \text{- - - - - -}$ to n terms.

Formula $S_n = \frac{a(1-r^n)}{1-r} \; |r| < 1$

Taking limit as n → ∞ on both sides

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} a\frac{(1-r^n)}{1-r}$$

$$= \lim_{n\to\infty} a\left[\frac{1}{1-r} - \frac{r^n}{1-r}\right]$$

$$= \lim_{n\to\infty}\left(\frac{a}{1-r}\right) - \lim_{n\to\infty}\frac{ar^n}{1-r}$$

as $n \to \infty, r^n \to 0$

Therefore $S\infty = \frac{a}{1-r} - 0$