Chapter 1
Quadratic Equations

1.1 Equation:
An equation is a statement of equality '=' between two expression for particular values of the variable. For example
$5x + 6 = 2$, x is the variable (unknown)
The equations can be divided into the following two kinds:

Conditional Equation:
It is an equation in which two algebraic expressions are equal for particular value/s of the variable e.g.,
a) $2x = 3$ is true only for $x = 3/2$
b) $x^2 + x - 6 = 0$ is true only for $x = 2, -3$

Note: for simplicity a conditional equation is called an equation.

Identity:
It is an equation which holds good for all value of the variable e.g;
a) $(a + b)\, x \equiv ax + bx$ is an identity and its two sides are equal for all values of x.
b) $(x + 3)(x + 4) \equiv x^2 + 7x + 12$ is also an identity which is true for all values of x.

For convenience, the symbol '=' shall be used both for equation and identity.

1.2 Degree of an Equation:
The degree of an equation is the highest sum of powers of the variables in one of the term of the equation. For example

$2x + 5 = 0$	1st degree equation in single variable
$3x + 7y = 8$	1st degree equation in two variables
$2x^2 - 7x + 8 = 0$	2nd degree equation in single variable
$2xy - 7x + 3y = 2$	2nd degree equation in two variables
$x^3 - 2x^2 + 7x + 4 = 0$	3rd degree equation in single variable
$x^2y + xy + x = 2$	3rd degree equation in two variables

1.3 Polynomial Equation of Degree n:
An equation of the form

$$a_nx^n + a_{n-1}x^{n-1} + \text{----------------} + a_3x^3 + a_2x^2 + a_1x + a_0 = 0 \text{--------------}(1)$$

Where n is a non-negative integer and a_n, a_{n-1}, -------------, a_3, a_2, a_1, a_0 are real constants, is called polynomial equation of degree n. Note that the degree of the equation in the single variable is the highest power of x which appear in the equation. Thus

$$3x^4 + 2x^3 + 7 = 0$$
$$x^4 + x^3 + x^2 + x + 1 = 0 \quad , \quad x^4 = 0$$

are all fourth-degree polynomial equations.

By the techniques of higher mathematics, it may be shown that nth degree equation of the form (1) has exactly n solutions (roots). These roots may be real, complex or a mixture of both. Further it may be shown that if such an equation has complex roots, they occur in pairs of conjugates complex numbers. In other words it cannot have an odd number of complex roots.

A number of the roots may be equal. Thus all four roots of $x^4 = 0$ are equal which are zero, and the four roots of $x^4 - 2x^2 + 1 = 0$ Comprise two pairs of equal roots (1, 1, -1, -1).

1.4 Linear and Cubic Equation:

The equation of first degree is **called linear equation.**

For example,

i) $x + 5 = 1$ (in single variable)

ii) $x + y = 4$ (in two variables)

The equation of third degree is **called cubic equation.**

For example,

i) $a_3x^3 + a_2x^2 + a_1x + a0 = 0$ (in single variable)

ii) $9x^3 + 5x^2 + 3x = 0$ (in single variable)

iii) $x^2y + xy + y = 8$ (in two variables)

1.5 Quadratic Equation:

The equation of second degree is called quadratic equation. The word quadratic comes from the Latin for "square", since the highest power of the unknown that appears in the equation is square. For example

$2x^2 – 3x + 7 = 0$ (in single variable)

$xy – 2x + y = 9$ (in two variable)

Standard form of quadratic equation

The standard form of the quadratic equation is $ax^2 + bx + c = 0$, where a, b and c are constants with $a \neq 0$.

If $b \neq 0$ then this equation is called **complete quadratic equation** in x.

If $b = 0$ then it is called a **pure or incomplete quadratic equation** in x.

For example, $5 x^2 + 6 x + 2 = 0$ is a complete quadratic equation is x.

and $3x^2 – 4 = 0$ is a pure or incomplete quadratic equation.

1.6 Roots of the Equation:

The value of the variable which satisfies the equation is called the root of the equation. A quadratic equation has two roots and hence there will be two values of the variable which satisfy the quadratic equation. For example the roots of $x^2 + x – 6 = 0$ are 2 and -3.

1.7 Methods of Solving Quadratic Equation:

There are three methods for solving a quadratic equation:

i) By factorization

ii) By completing the square

iii) By using quadratic formula

i) Solution by Factorization:

Method:

Step I: Write the equation in standard form.

Step II: Factorize the quadratic equation on the left hand side if possible.

Step III: The left hand side will be the product of two linear factors. Then equate each of the linear factor to zero and solve for values of x. These values of x give the solution of the equation.

Example 1:

Solve the equation $3x^2 + 5x = 2$

Solution:

$$3x^2 + 5x = 2$$

Write in standard form $3x^2 + 5x – 2 = 0$

Factorize the left hand side $3x^2 + 6x - x - 2 = 0$

$$3x(x + 2) - 1(x + 2) = 0$$

$$(3x - 1)(x + 2) = 0$$

Equate each of the linear factor to zero.

$3x - 1 = 0$ or $x + 2 = 0$

$3x = 1$ or $x = -2$

$x = \frac{1}{3}$

$x = \frac{1}{3}$, - 2 are the roots of the Equation.

Solution Set = $\{\frac{1}{3}, -2\}$

Example 2:

Solve the equation $6x^2 - 5x = 4$

Solution:

$6x^2 - 5x = 4$

$6x^2 - 5x - 4 = 0$

$6x^2 - 8x + 3x - 4 = 0$

$2x(3x - 4) + 1(3x - 4) = 0$

$(2x + 1)(3x - 4) = 0$

$\therefore$ Either $2x + 1 = 0$ or $3x - 4 = 0$

Which gives $2x = -1$ which gives $3x = 4$

$\Rightarrow x = -\frac{1}{2}$ $\Rightarrow x = \frac{4}{3}$

$\therefore$ Required Solution Set = $\left\{-\frac{1}{2}, \frac{4}{3}\right\}$

ii) Solution of quadratic equation by Completing the Square

Method:

Step I: Write the quadratic equation is standard form.

Step II: Divide both sides of the equation by the co-efficient of x^2 if it is not already 1.

Step III: Shift the constant term to the R.H.S.

Step IV: Add the square of one-half of the co-efficient of x to both side.

Step V: Write the L.H.S as complete square and simplify the R.H.S.

Step VI: Take the square root on both sides and solve for x.

Example 3:

Solve the equation $3x^2 = 15 - 4x$ by completing the square.

Solution: $3x^2 = 15 - 4x$

Step I Write in standard form: $3x^2 + 4x - 15 = 0$

Step II Dividing by 3 to both sides: $x^2 + \frac{4}{3}x - 5 = 0$

Step III Shift constant term to R.H.S: $x^2 + \frac{4}{3}x = 5$

Step IV Adding the square of one half of the co-efficient of x. i.e., $\left(\frac{4}{6}\right)^2$ on both sides:

$$x^2 + \frac{4}{3}x + \left(\frac{4}{6}\right)^2 = 5 + \left(\frac{4}{6}\right)^2$$

Step V: Write the L.H.S. as complete square and simplify the R.H.S.
:

$$\left(x + \frac{4}{6}\right)^2 = 5 + \frac{16}{36}$$

$$= \frac{180 + 16}{36}$$

$$\left(x + \frac{4}{6}\right)^2 = \frac{196}{36}$$

Step VI: Taking square root of both sides and Solve for x

$$\sqrt{\left(x + \frac{4}{6}\right)^2} = \sqrt{\frac{196}{36}}$$

$$x + \frac{4}{6} = \pm\frac{14}{6}$$

$$x + \frac{4}{6} = \pm\frac{7}{3}$$

$$x + \frac{4}{6} = \frac{7}{3}, \qquad x + \frac{4}{6} = -\frac{7}{3}$$

$$\Rightarrow \quad x = \frac{7}{3} - \frac{4}{6} \qquad \Rightarrow \quad x = -\frac{7}{3} - \frac{4}{6}$$

$$x = \frac{10}{6}, \qquad x = \frac{-18}{6}$$

$$x = \frac{5}{3}, \qquad x = -3$$

Hence, the solution set = $\left\{-3, \frac{5}{3}\right\}$

Example 4:

Solve the equation $a^2 x^2 = ab\,x + 2b^2$ by completing the square.

Solution:

$$a^2 x^2 = ab\,x + 2b^2$$

$$a^2 x^2 - ab\,x - 2b^2 = 0$$

Dividing both sides by a^2, we have

$$x^2 - \frac{bx}{a} - \frac{2b^2}{a^2} = 0$$

$$x^2 - \frac{bx}{a} = \frac{2b^2}{a^2}$$

Adding the square of one half of the co-efficient of x i.e., $\left(-\frac{b}{2a}\right)^2$ on both sides.

$$x^2 - \frac{bx}{a} + \left(-\frac{b}{2a}\right)^2 = \frac{2b^2}{a^2} + \left(-\frac{b}{2a}\right)^2$$

$$\left(x - \frac{b}{2a}\right)^2 = \frac{2b^2}{a^2} + \frac{b^2}{4a^2}$$

$$\left(x - \frac{b}{2a}\right)^2 = \frac{8b^2 + b^2}{4a^2}$$

$$\left(x - \frac{b}{2a}\right)^2 = \frac{9b^2}{4a^2}$$

Taking square root on both sides

$$x - \frac{b}{2a} = \pm\frac{3b}{2a}$$

$x - \frac{b}{2a} = \frac{3b}{2a}$	$x - \frac{b}{2a} = \frac{3b}{2a}$
$\Rightarrow x = \frac{b}{2a} + \frac{3b}{2a}$	$\Rightarrow x = \frac{b}{2a} - \frac{3b}{2a}$
$\Rightarrow x = \frac{b + 3b}{2a}$	$\Rightarrow x = \frac{b - 3b}{2a}$
$\Rightarrow x = \frac{4b}{2a}$	$\Rightarrow x = -\frac{2b}{2a}$
$\Rightarrow x = \frac{2b}{a}$	$\Rightarrow x = -\frac{b}{a}$

Solution Set = $\left\{\frac{2b}{a}, -\frac{b}{a}\right\}$

iii) Derivation of Quadratic formula

Consider the standard form of quadratic equation $ax^2 + bx + c = 0$.
Solve this equation by completing the square.

$$ax^2 + bx + c = 0$$

Dividing both sides by a

$$x^2 + \frac{b}{a}x + \frac{c}{a} = o$$

Take the constant term to the R.H.S

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

To complete the square on L.H.S. add $\left(\frac{b}{2a}\right)^2$ to both sides.

$$X^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Taking square root of both sides

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = -\frac{b}{2a} \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x = -\frac{b}{2a} \pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$\boxed{x = \frac{-b \pm\sqrt{b^2 - 4ac}}{2a}}$$

which is called the **Quadratic formula.**

Where, a = co-efficient of x^2 , b = coefficient of x , c = constant term

Actually, the Quadratic formula is the general solution of the quadratic equation $ax^2 + b x + c = 0$

Note: $\frac{-b \pm\sqrt{b^2 - 4ac}}{2a}$, $\frac{-b -\sqrt{b^2 - 4ac}}{2a}$ are also called roots of the quadratic equation

Method:

To solve the quadratic equation by Using Quadratic formula:

Step I: Write the Quadratic Equation in Standard form.

Step II: By comparing this equation with standard form $ax^2 + b x + c = 0$ to identify the values of a , b , c.

Step III: Putting these values of a, b, c in Quadratic formula

$$x = \frac{-b \pm\sqrt{b^2 - 4ac}}{2a}$$ and solve for x.

Example 5:

Solve the equation $3x^2 + 5x = 2$

Solution:

$$3x^2 + 5x = 2$$
$$3x^2 + 5x - 2 = 0$$

Composing with the standard form $ax^2 + bx + c = 0$, we have $a = 3$, $b = 5$, $c = -2$.

Putting these values in Quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-5 \pm \sqrt{(5)^2 - 4(3)(-2)}}{2(3)}$$

$$= \frac{-5 \pm \sqrt{25 + 24}}{6}$$

$$x = \frac{-5 \pm 7}{6}$$

$$x = \frac{-5 + 7}{6} \qquad x = \frac{-5 - 7}{6}$$

$$x = \frac{2}{6} \qquad x = \frac{-12}{6}$$

$$x = \frac{1}{3} \qquad x = -2$$

$$\text{Sol. Set} = \left\{\frac{1}{3},\ 2\right\}$$

Example 6:

Solve the equation $15x^2 - 2ax - a^2 = 0$ by using Quadratic formula:

Solution:

$15x^2 - 2ax - a^2 = 0$

Comparing this equation with General Quadratic Equation

Here, $a = 15$, $b = -2a$, $c = -a^2$

Putting these values in Quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-2a) \pm \sqrt{(-2a)^2 - 4(15)(-a^2)}}{2(15)}$$

$$= \frac{-(-2a) \pm \sqrt{4a^2 + 60a^2}}{30}$$

$$= \frac{2a \pm 8a}{30}$$

$$x = \frac{2a + 8a}{30} \qquad x = \frac{2a - 8a}{30}$$

$$x = \frac{10a}{30} \qquad x = \frac{-6a}{30}$$

$$x = \frac{a}{3} \qquad x = -\frac{a}{5}$$

$$\text{Sol. Set} = \left\{\frac{a}{3},\ -\frac{a}{5}\right\}$$

Example 7:

Solve the equation $\frac{1}{2x-5} + \frac{5}{2x-1} = 2$ by using Quadratic formula.

Solution:

$$\frac{1}{2x-5} + \frac{5}{2x-1} = 2$$

Multiplying throughout by (2x – 5)(2x – 1), we get

$$(2x-1) + 5(2x-5) = 2(2x-5)(2x-1)$$
$$2x - 1 + 10x - 25 = 8x^2 - 24x + 10$$
$$8x^2 - 36x + 36 = 0$$
$$2x^2 - 9x + 9 = 0$$

Comparing this equation with General Quadratic Equation

Here, $a = 2$, $b = -9$, $c = 9$

Putting these values in the Quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{(-9) \pm \sqrt{(-9)^2 - 4(2)(9)}}{2(2)}$$
$$= \frac{9 \pm \sqrt{81 - 72}}{4}$$
$$= \frac{9 \pm 3}{4}$$

$$x = \frac{9+3}{4} \qquad x = \frac{9-3}{4}$$
$$x = \frac{12}{4} \qquad x = \frac{6}{4}$$
$$x = 3 \qquad x = -\frac{3}{2}$$

Sol. Set $\left\{3, -\frac{3}{2}\right\}$

Exercise 1.1

Q.1. Solve the following equations by factorization.

(i). $x^2 + 7x = 8$

(ii). $3x^2 + 7x + 4 = 0$

(iii). $x^2 - 3x = 2x - 6$

(iv). $3x^2 - 1 = \frac{1}{5}(1 - x)$

(v). $(2x + 3)(x + 1) = 1$

(vi). $\frac{1}{2x-5} + \frac{5}{2x-1} = 2$

(vii). $\frac{4}{x-1} - \frac{5}{x+2} = \frac{3}{x}$ (viii). $\frac{1}{a+b+x} = \frac{1}{a} + \frac{1}{b} + \frac{1}{x}$

(ix). $abx^2 + (b^2 - ac)x - bc = 0$ (x). $(a+b)x^2 + (a+2b+c)x + (b+c) = 0$

(xi). $\frac{a}{ax-1} + \frac{b}{bx-1} = a + b$ (xii). $\frac{x+2}{x-1} + 2\frac{2}{3} = \frac{x+3}{x-2}$

Q.2. Solve the following equations by the method of completing the square.

(i). $x^2 - 6x + 8 = 0$ (ii). $32 - 3x^2 = 10x$

(iii). $(x-2)(x+3) = 2(x+11)$ (iv). $x^2 + (a+b)x + ab = 0$

(v). $x + \frac{1}{x} = \frac{10}{3}$ (vi). $\frac{10}{x-5} + \frac{10}{x+5} = \frac{5}{6}$

(vii). $2x^2 - 5bx = 3b^2$ (viii). $x^2 - 2ax + a^2 - b^2 = 0$

Q.3 Solve the following equations by using quadratic formula.

(i). $2x^2 + 3x - 9 = 0$ (ii). $(x+1)^2 = 3x + 14$

(iii). $\frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3} = \frac{3}{x}$ (iv). $x^2 - 3\left(x + \frac{25}{4}\right) = 9x - \frac{25}{2}$

(v). $x^2 + (m-n)x - 2(m-n)^2 = 0$ (vi) $mx^2 + (1+m)x + 1 = 0$

(vii) $abx^2 + (2b-3a)x - 6 = 0$ (viii). $x^2 + (b-a)x - ab = 0$

(ix) $\frac{x}{x+1} + \frac{x+1}{x+2} + \frac{x+2}{x+3} = 3$

Q.4 The sum of a number and its square is 56 . Find the number.

Q.5 A projectile is fired vertically into the air. The distance (in meter) above the ground as a function of time (in seconds) is given by $s = 300 - 100t - 16t^2$. When will the projectile hit the ground?

Q.6 The hypotenuse of a right triangle is 18 meters. If one side is 4 meters longer than the other side, what is the length of the shorter side ?

Answers 1.1

Q.1. (i). {1, -8} (ii). $\left\{-1, \frac{-4}{3}\right\}$ (iii). {2, 3}

(iv). $\left\{\frac{3}{5}, -\frac{2}{3}\right\}$ (v). $\left\{-2, -\frac{1}{2}\right\}$ (vi). $\left\{-\frac{3}{2}, 3\right\}$

(vii). $\left\{-\frac{1}{2}, 3\right\}$ (viii). {-a, -b} (ix). $\{-\frac{b}{a}, \frac{c}{b}\}$

(x). $\{-1, -\frac{b+c}{a+b}\}$ (xi). $\{\frac{a+b}{ab}, \frac{2}{a+b}\}$ (xii). $\{\frac{13}{4}, \frac{1}{2}\}$

Q.2. (i). {2, 4} (ii). $\{2, -\frac{16}{3}\}$ (iii). $\{\frac{1\pm\sqrt{113}}{2}\}$

(iv). $\{-a, -b\}$ (v). $\{3, \frac{1}{3}\}$ (vi). {-1, 25}

(vii). $\{3b, -\frac{b}{2}\}$ (viii) $\{(a+b), (a-b)\}$

Q.3. (i). $\{\frac{3}{2}, -3\}$ (ii). $\{\frac{1\pm\sqrt{53}}{2}\}$ (iii). $\{\frac{-11\pm\sqrt{13}}{6}\}$

(iv). $\{\frac{25}{2}, -\frac{1}{2}\}$ (v). $\{m-n, -2(m-n)\}$ (vi) $\left\{-1, \frac{-1}{m}\right\}$

(vii). $\{-\frac{2}{a}, \frac{3}{b}\}$ (viii). $\{-b, a\}$ (ix) $\left\{\frac{-6+\sqrt{3}}{3}, \frac{-6-\sqrt{3}}{3}\right\}$

Q.4. 7 , - 8 Q.5. 8.465 seconds Q.6. 10.6 m

1.8 Classification of Numbers

1. The Set N of Natural Numbers:

Whose elements are the counting, or natural numbers:

N = {1, 2, 3, - - - - - - - - }

2. The Set Z of Integers:

Whose elements are the positive and negative whole numbers and zero:

Z = {- - - - - - , -2, -1, 0, 1, 2, - - - - - - }

3. The Set Q of Rational Numbers: Whose elements are all those numbers that can be represented as the quotient of two integers $\frac{a}{b}$, where $b \neq 0$. Among the elements of Q are such numbers as $-\frac{3}{4}, \frac{18}{27}, \frac{5}{1}, -\frac{9}{1}$. In symbol

$$Q = \left\{\frac{a}{b} \mid a, b \in Z, b \neq 0\right\}$$

Equivalently, rational numbers are numbers with terminating or repeating decimal representation, such as

1.125, 1.52222, 1.56666, 0.3333