TAHIR MEHMOOD
M.Sc Math
0345-6510779

Binomial Theorem:-

"A rule for the expansion of Binomial expression having "n" index is called Binomial Theorem." For any +ve integer n

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \dots + \binom{n}{r}a^{n-r}b^r + \dots + \binom{n}{n}b^n$$

Breifly, it can be written as $(a+b)^n = \sum_{r=0}^{n}\binom{n}{r}a^{n-r}b^r$.

Proof:- We will prove it by Mathematical induction.

Let $n=1$ then $(a+b)^1 = \binom{1}{0}a^1 + \binom{1}{1}b^1 = a+b$ which is true

Condition (i) holds as $\binom{1}{0} = \binom{1}{1} = 1$

Suppose result holds for $n=K$

$$(a+b)^K = \binom{K}{0}a^K + \binom{K}{1}a^{K-1}b + \dots + \binom{K}{r}a^{K-r}b^r + \dots + \binom{K}{K}b^K \quad \text{--- (1)}$$

Now we want to prove it for $n=K+1$

Multiplying both sides by $(a+b)$, we have:

$$(a+b)\cdot(a+b)^K = (a+b)\cdot\left[\binom{K}{0}a^K + \binom{K}{1}a^{K-1}b + \binom{K}{2}a^{K-2}b^2 + \dots + \binom{K}{r}a^{K-r}b^r + \dots + \binom{K}{K}b^K\right]$$

$$(a+b)^{K+1} = a\left[\binom{K}{0}a^K + \binom{K}{1}a^{K-1}b + \dots + \binom{K}{r}a^{K-r}b^r + \dots + \binom{K}{K}b^K\right]$$
$$+ b\left[\binom{K}{0}a^K + \binom{K}{1}a^{K-1}b + \dots + \binom{K}{r}a^{K-r}b^r + \dots + \binom{K}{K}b^K\right]$$

$$(a+b)^{K+1} = \binom{K}{0}a^{K+1} + \left[\binom{K}{1} + \binom{K}{0}\right]a^K b + \left[\binom{K}{2} + \binom{K}{1}\right]a^{K-1}b^2 + \dots + \left[\binom{K}{K-1} + \binom{K}{K}\right]ab^K + \binom{K}{K}b^{K+1}$$

$\because \binom{K}{0} = \binom{K+1}{0} = 1$, $\binom{K}{K} = \binom{K+1}{K+1} = 1$, $\binom{K}{r} + \binom{K}{r-1} = \binom{K+1}{r}$

So

$$(a+b)^{K+1} = \binom{K+1}{0}a^{K+1} + \binom{K+1}{1}a^K b + \dots + \binom{K+1}{r}a^{K+1-r}b^r + \dots + \binom{K+1}{K+1}b^{K+1}$$

condition (ii) holds so it holds for all +ve integers. (Proved)

In this theorem, $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \dots, \binom{n}{n}$ are called Binomial Coefficients.

Charactristics of Binomial Theorem:-

i) Number of terms are one more than index in each binomial expression.
ii) Sum of exponents of "a" and "b" is equal to index in each term.
iii) The exponent of "a" decreases from index to zero.
iv) The exponent of "b" increases from zero to index.
v) Coefficient of each term from beginning to end is equidistant as $\binom{n}{r} = \binom{n}{n-r}$.
vi) The general term is $(r+1)$th term denoted as

$$T_{r+1} = \binom{n}{r}a^{n-r}b^r \quad \text{where } r = 0, 1, 2, \dots, n$$

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vii) Sum of all binomial Coefficients is 2^n in $(a+b)^n$.

i.e. $\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}=2^n$

viii) If in $(a+b)^n$, n is even then $\frac{n+2}{2}$ is middle term

if "n" is odd then $\frac{n+1}{2}$ and $\frac{n+3}{2}$ are middle terms.

EXERCISE No: 8.2

Q.1 Use Binomial theorem, expand the followings:

i) $(a+2b)^5$

$=\binom{5}{0}a^5+\binom{5}{1}a^4(2b)^1+\binom{5}{2}a^3(2b)^2+\binom{5}{3}a^2(2b)^3+\binom{5}{4}a^1(2b)^4+\binom{5}{5}(2b)^5$

$=(1)a^5+(5)(a^4)(2b)+(10)a^3(4b^2)+(10)a^2(8b^3)+(5)(a)(16b^4)+(1)(32b^5)$

$=a^5+10a^4b+40a^3b^2+80a^2b^3+80ab^4+32b^5$ Ans.

ii) $\left(\frac{x}{2}-\frac{2}{x^2}\right)^6$

$=\binom{6}{0}\left(\frac{x}{2}\right)^6+\binom{6}{1}\left(\frac{x}{2}\right)^5\left(-\frac{2}{x^2}\right)+\binom{6}{2}\left(\frac{x}{2}\right)^4\left(-\frac{2}{x^2}\right)^2+\binom{6}{3}\left(\frac{x}{2}\right)^3\left(-\frac{2}{x^2}\right)^3+\binom{6}{4}\left(\frac{x}{2}\right)^2\left(-\frac{2}{x^2}\right)^4$
$+\binom{6}{5}\left(\frac{x}{2}\right)\left(-\frac{2}{x^2}\right)^5+\binom{6}{6}\left(-\frac{2}{x^2}\right)^6$

$=(1)\left(\frac{x^6}{64}\right)+6\left(\frac{x^5}{32}\right)\left(-\frac{2}{x^2}\right)+15\left(\frac{x^4}{16}\right)\left(\frac{4}{x^4}\right)+20\left(\frac{x^3}{8}\right)\left(\frac{-8}{x^6}\right)+15\left(\frac{x^2}{4}\right)\left(\frac{16}{x^8}\right)+6\left(\frac{x}{2}\right)\left(-\frac{32}{x^{10}}\right)$ $+1\cdot\frac{64}{x^{12}}$

$=\frac{x^6}{64}-\frac{3x^3}{8}+\frac{15}{4}-\frac{20}{x^3}+\frac{60}{x^6}-\frac{96}{x^9}+\frac{64}{x^{12}}$ Ans.

iii) $\left(3a-\frac{x}{3a}\right)^4$

$=\binom{4}{0}(3a)^4+\binom{4}{1}(3a)^3\left(-\frac{x}{3a}\right)^1+\binom{4}{2}(3a)^2\left(-\frac{x}{3a}\right)^2+\binom{4}{3}(3a)^1\left(-\frac{x}{3a}\right)^3+\binom{4}{4}\left(-\frac{x}{3a}\right)^4$

$=1(81a^4)+4(27a^3)\left(\frac{-x}{3a}\right)+6(9a^2)\left(\frac{x^2}{9a^2}\right)+4(3a)\left(-\frac{x^3}{27a^3}\right)+1\left(\frac{x^4}{81a^4}\right)$

$=81a^4-36a^2x+6x^2-\frac{4x^3}{9a^2}+\frac{x^4}{81a^4}$ Ans.

iv) $\left(2a-\frac{x^2}{a}\right)^7$

$=\binom{7}{0}(2a)^7+\binom{7}{1}(2a)^6\left(-\frac{x^2}{a}\right)+\binom{7}{2}(2a)^5\left(-\frac{x^2}{a}\right)^2+\binom{7}{3}(2a)^4\left(-\frac{x^2}{a}\right)^3+\binom{7}{4}(2a)^3\left(-\frac{x^2}{a}\right)^4$
$+\binom{7}{5}(2a)^2\left(-\frac{x^2}{a}\right)^5+\binom{7}{6}(2a)\left(-\frac{x^2}{a}\right)^6+\binom{7}{7}\left(-\frac{x^2}{a}\right)^7$

$=1(128a^7)+7(64a^6)\left(-\frac{x^2}{a}\right)+21(32a^5)\left(\frac{x^4}{a^2}\right)+35(16a^4)\left(-\frac{x^6}{a^3}\right)+35(8a^3)\left(\frac{x^8}{a^4}\right)$
$+21(4a^2)\left(-\frac{x^{10}}{a^5}\right)+7(2a)\left(\frac{x^{12}}{a^6}\right)+1\left(-\frac{x^{14}}{a^7}\right)$

$=128a^7-448a^5x^2+672a^3x^4-560ax^6+280\frac{x^8}{a}-84\frac{x^{10}}{a^3}+14\frac{x^{12}}{a^5}-\frac{x^{14}}{a^7}$ Ans.

v) $\left(\frac{x}{2y}-\frac{2y}{x}\right)^8$

$$=\binom{8}{0}\left(\frac{x}{2y}\right)^8+\binom{8}{1}\left(\frac{x}{2y}\right)^7\left(-\frac{2y}{x}\right)+\binom{8}{2}\left(\frac{x}{2y}\right)^6\left(-\frac{2y}{x}\right)^2+\binom{8}{3}\left(\frac{x}{2y}\right)^5\left(-\frac{2y}{x}\right)^3+\binom{8}{4}\left(\frac{x}{2y}\right)^4\left(-\frac{2y}{x}\right)^4$$

$$+\binom{8}{5}\left(\frac{x}{2y}\right)^3\left(-\frac{2y}{x}\right)^5+\binom{8}{6}\left(\frac{x}{2y}\right)^2\left(-\frac{2y}{x}\right)^6+\binom{8}{7}\left(\frac{x}{2y}\right)^1\left(-\frac{2y}{x}\right)^7+\binom{8}{8}\left(-\frac{2y}{x}\right)^8$$

$$=1\left(\frac{x^8}{256y^8}\right)+8\left(\frac{x^7}{128y^7}\right)\left(-\frac{2y}{x}\right)+28\left(\frac{x^6}{64y^6}\right)\left(\frac{4y^2}{x^2}\right)+56\left(\frac{x^5}{32y^5}\right)\left(-\frac{8y^3}{x^3}\right)+70\left(\frac{x^4}{16y^4}\right)\left(\frac{16y^4}{x^4}\right)$$

$$+56\left(\frac{x^3}{8y^3}\right)\left(-\frac{32y^5}{x^5}\right)+28\left(\frac{x^2}{4y^2}\right)\left(\frac{64y^6}{x^6}\right)+8\left(\frac{x}{2y}\right)\left(-\frac{128y^7}{x^7}\right)+1\left(\frac{256y^8}{x^8}\right)$$

$$=\frac{x^8}{256y^8}-\frac{x^6}{8y^6}+\frac{7x^4}{4y^4}-\frac{14x^2}{y^2}+70-224\frac{y^2}{x^2}+448\frac{y^4}{x^4}-512\frac{y^6}{x^6}+\frac{256y^8}{x^8}$$ Ans.

vi) $\left(\sqrt{\frac{a}{x}}-\sqrt{\frac{x}{a}}\right)^6$

$$=\binom{6}{0}\left(\sqrt{\frac{a}{x}}\right)^6+\binom{6}{1}\left(\sqrt{\frac{a}{x}}\right)^5\left(-\sqrt{\frac{x}{a}}\right)^1+\binom{6}{2}\left(\sqrt{\frac{a}{x}}\right)^4\left(-\sqrt{\frac{x}{a}}\right)^2+\binom{6}{3}\left(\sqrt{\frac{a}{x}}\right)^3\left(-\sqrt{\frac{x}{6}}\right)^3$$

$$+\binom{6}{4}\left(\sqrt{\frac{a}{x}}\right)^2\left(-\sqrt{\frac{x}{a}}\right)^4+\binom{6}{5}\left(\sqrt{a/x}\right)\left(-\sqrt{x/a}\right)^5+\binom{6}{6}\left(-\sqrt{\frac{x}{a}}\right)^6$$

$$=1\left(\frac{a^3}{x^3}\right)+6\left(\frac{a^2\sqrt{a}}{x^2\sqrt{x}}\right)\left(-\frac{\sqrt{x}}{\sqrt{a}}\right)+15\left(\frac{a^2}{x^2}\right)\left(\frac{x}{a}\right)+20\left(\frac{a\sqrt{a}}{x\sqrt{x}}\right)\left(-\frac{x\sqrt{x}}{a\sqrt{a}}\right)+15\left(\frac{a}{x}\right)\left(+\frac{x^2}{a^2}\right)$$

$$+6\left(\frac{\sqrt{a}}{\sqrt{x}}\right)\left(-\frac{x^2\sqrt{x}}{a^2\sqrt{a}}\right)+1.\left(\frac{x^3}{a^3}\right)$$

$$=\frac{a^3}{x^3}-6\frac{a^2}{x^2}+15\frac{a}{x}-20+15\frac{x}{a}-6\frac{x^2}{a^2}+\frac{x^3}{a^3}$$ Ans.

Q.2 Calculate the following by means of Binomial Theorem:

i) $(0.97)^3=(1-0.03)^3$

$$=\binom{3}{0}(1)^3+\binom{3}{1}(1)^2(-0.03)+\binom{3}{2}(1)^1(-0.03)^2+\binom{3}{3}(-0.03)^3$$

$$=1.(1)+3(1)(-0.03)+3(1)(0.0009)+1(-0.000027)$$

$$=1-0.09+0.0027-0.000027=0.912673$$ Ans.

ii) $(2.02)^4=(2+0.02)^4$

$$=\binom{4}{0}(2)^4+\binom{4}{1}(2)^3(0.02)+\binom{4}{2}(2)^2(0.02)^2+\binom{4}{3}(2)^1(0.02)^3+\binom{4}{4}(0.02)^4$$

$$=1.(16)+4(8)(0.02)+6(4)(0.0004)+4(2)(0.000008)+1.(0.00000016)$$

$$=16+0.64+0.0096+0.000064+0.00000016=16.649664$$ Ans.

iii) $(9.98)^4=(10-0.02)^4$

$$=\binom{4}{0}(10)^4+\binom{4}{1}(10)^3(-0.02)+\binom{4}{2}(10)^2(-0.02)^2+\binom{4}{3}(10)(-0.02)^3+\binom{4}{4}(-0.02)^4$$

$$=1(10000)+4(1000)(-0.02)+6(100)(0.0004)+4(10)(-0.000008)+1.(0.00000016)$$

$$=10000-80+0.24-0.00032+0.00000016=9920.23968016$$ Ans.

iv) $(2.1)^5 = (2+0.1)^5$

$= \binom{5}{0}(2)^5 + \binom{5}{1}(2)^4(0.1) + \binom{5}{2}(2)^3(0.1)^2 + \binom{5}{3}(2)^2(0.1)^3 + \binom{5}{4}(2)(0.1)^4 + \binom{5}{5}(0.1)^5$

$= 1(32) + 5(16)(0.1) + 10(8)(0.01) + 10(4)(0.001) + 5(2)(0.0001) + 1(0.00001)$

$= 32 + 8 + 0.8 + 0.04 + 0.0001 + 0.00001 = 40.84011$ Ans

Q.3 Expand and Simplify the followings:

i) $(a+\sqrt{2}x)^4 + (a-\sqrt{2}x)^4$

$= \left[\binom{4}{0}a^4 + \cancel{\binom{4}{1}a^3(\sqrt{2}x)} + \binom{4}{2}a^2(\sqrt{2}x)^2 + \binom{4}{3}\cancel{a(\sqrt{2}x)^3} + \binom{4}{4}(\sqrt{2}x)^4\right]$

$+ \left[\binom{4}{0}a^4 + \binom{4}{1}\cancel{a^3(-\sqrt{2}x)} + \binom{4}{2}a^2(-\sqrt{2}x)^2 + \cancel{\binom{4}{3}a(-\sqrt{2}x)^3} + \binom{4}{4}(-\sqrt{2}x)^4\right]$

$= 2\left[\binom{4}{0}a^4 + \binom{4}{2}a^2(2x^2) + \binom{4}{4}(4x^4)\right]$

$= 2\left[1.a^4 + 6a^2(2x^2) + 1.4x^4\right] = 2a^4 + 24a^2x^2 + 8x^4$ Ans.

ii) $(2+\sqrt{3})^5 + (2-\sqrt{3})^5$

$= \left[\binom{5}{0}2^5 + \cancel{\binom{5}{1}2^4(\sqrt{3})} + \binom{5}{2}2^3(\sqrt{3})^2 + \cancel{\binom{5}{3}2^2.(\sqrt{3})^3} + \binom{5}{4}2(\sqrt{3})^4 + \cancel{\binom{5}{5}(\sqrt{3})^5}\right]$

$+ \left[\binom{5}{0}2^5 + \cancel{\binom{5}{1}2^4(-\sqrt{3})} + \binom{5}{2}2^3(-\sqrt{3})^2 + \cancel{\binom{5}{3}2^2(-\sqrt{3})^3} + \binom{5}{4}2(-\sqrt{3})^4 + \cancel{\binom{5}{5}(-\sqrt{3})^5}\right]$

$= 2\left[\binom{5}{0}(32) + \binom{5}{2}(8)(3) + \binom{5}{4}(2)(9)\right] = 2\left[1.(32) + (10)(8)(3) + (5)(2)(9)\right]$

$= 2[32 + 240 + 90] = 2[362] = 724$ Ans.

iii) $(2+i)^5 - (2-i)^5$

$= \left[\binom{5}{0}2^5 + \binom{5}{1}2^4.i + \binom{5}{2}2^3.i^2 + \binom{5}{3}2^2.i^3 + \binom{5}{4}2.i^4 + \binom{5}{5}i^5\right] - \left[\binom{5}{0}2^5 + \right.$

$\left. + \binom{5}{1}2^4(-i) + \binom{5}{2}2^3(-i)^2 + \binom{5}{3}2^2(-i)^3 + \binom{5}{4}2(-i)^4 + \binom{5}{5}(-i)^5\right]$

$= \cancel{\binom{5}{0}2^5} + \binom{5}{1}2^4i + \cancel{\binom{5}{2}2^3i^2} + \binom{5}{3}2^2i^3 + \cancel{\binom{5}{4}2.i^4} + \binom{5}{5}i^5 - \cancel{\binom{5}{0}2^5} + \binom{5}{1}2^4i$

$- \cancel{\binom{5}{2}2^3i^2} + \binom{5}{3}2^2i^3 - \cancel{\binom{5}{4}2i^4} + \binom{5}{5}i^5$ $\quad \because i^3 = -i$

$= 2\left[\binom{5}{1}2^4i + \binom{5}{3}2^2i^3 + \binom{5}{5}i^5\right] = 2\left[5(16)i + 10(4)(-i) + 1(i)\right]$ $\quad \because i^5 = i$

$= 2[80i - 40i + i] = 2[41i] = 82i$ Ans.

iv) $(x+\sqrt{x^2-1})^3 + (x-\sqrt{x^2-1})^3$

$= x^3 + \cancel{3x^2\sqrt{x^2-1}} + 3x(\sqrt{x^2-1})^2 + \cancel{(\sqrt{x^2-1})^3} + x^3 - \cancel{3x\sqrt{x^2-1}} + 3x(\sqrt{x^2-1})^2$

$- \cancel{(\sqrt{x^2-1})^3}$

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$= 2x^3 + 6x(x^2-1)$

$= 2x^3 + 6x^3 - 6x = 8x^3 - 6x$ Ans

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Q.4 Expand in ascending Power of x:

i) $(2+x-x^2)^4 = [(2+x) - x^2]^4$

$= \left[\binom{4}{0}(x+2)^4 + \binom{4}{1}(x+2)^3(-x^2) + \binom{4}{2}(x+2)^2(-x^2)^2 + \binom{4}{3}(x+2)(-x^2)^3 + \binom{4}{4}(-x^2)^4\right]$

$= 1(x^2+4x+4)^2 + 4(x^3+6x^2+12x+8)(-x^2) + 6(x^2+4x+4)(x^4) + 4(x+2)(-x^6) + 1(x^8)$

$= x^4 + 16x^2 + 16 + 8x^3 + 32x + 8x^2 - 4x^5 - \cancel{24x^4} - 48x^3 - 32x^2 + 6x^6 + 24x^5 + \cancel{24x^4}$
$- 4x^7 - 8x^6 + x^8$

$= 16 + 32x - 8x^2 - 40x^3 + x^4 + 20x^5 - 2x^6 - 4x^7 + x^8$ Ans.

ii) $(1-x+x^2)^4 = [(1-x) + x^2]^4$

$= \left[\binom{4}{0}(1-x)^4 + \binom{4}{1}(1-x)^3 \cdot x^2 + \binom{4}{2}(1-x)^2 \cdot x^4 + \binom{4}{3}(1-x)x^6 + \binom{4}{4}x^8\right]$

$= 1.(1+x^2-2x)^2 + 4(1-3x+3x^2-x^3)x^2 + 6(1+x^2-2x)x^4 + 4(1-x)x^6 + 1.x^8$

$= 1 + x^4 + 4x^2 + 2x^2 - 4x^3 - 4x + 4x^2 - 12x^3 + 12x^4 - 4x^5 + 6x^4 + 6x^6 - 12x^5 + 4x^6 - 4x^7 + x^8$

$= 1 - 4x + 10x^2 - 16x^3 + 19x^4 - 16x^5 + 10x^6 - 4x^7 + x^8$ Ans.

iii) $(1-x-x^2)^4 = [(1-x) - x^2]^4$

$= \left[\binom{4}{0}(1-x)^4 + \binom{4}{1}(1-x)^3(-x^2) + \binom{4}{2}(1-x)^2(-x^2)^2 + \binom{4}{3}(1-x)(-x^2)^3 + \binom{4}{4}(-x^2)^4\right]$

$= [1.(x^2+1-2x)^2 - 4(1-x^3-3x+3x^2)x^2 + 6(1+x^2-2x)(x^4) - 4(1-x)x^6 + 1.x^8]$

$= x^4 + 1 + 4x^2 + 2x^2 - 4x - 4x^3 - 4x^2 + 4x^5 + 12x^3 - 12x^4 + 6x^4 + 6x^6 - 12x^5 - 4x^6 + 4x^7 + x^8$

$= 1 - 4x + 2x^2 + 8x^3 - 5x^4 - 8x^5 + 2x^6 + 4x^7 + x^8$ Ans.

Q.5 Expand in descending power of x:

i) $(x^2+x-1)^3 = [(x^2-1) + x]^3$

$= (x^2-1)^3 + 3(x^2-1)^2 \cdot x + 3(x^2-1)x^2 + x^3$

$= x^6 - 3x^4 + 3x^2 - 1 + 3(x^4+1-2x^2)x + 3x^4 - 3x^2 + x^3$

$= x^6 - \cancel{3x^4} + \cancel{3x^2} - 1 + 3x^5 + 3x - 6x^3 + \cancel{3x^4} - \cancel{3x^2} + x^3$

$= x^6 + 3x^5 - 5x^3 + 3x - 1$ Ans.

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ii) $(x-1-\frac{1}{x})^3 = \left[(x-1)-\frac{1}{x}\right]^3$

$= (x-1)^3 - 3(x-1)^2 \cdot \frac{1}{x} + 3(x-1)\frac{1}{x^2} - \frac{1}{x^3}$

$= x^3 - 3x^2 + 3x - 1 - \frac{3}{x}\left[x^2 - 2x + 1\right] + \frac{3}{x^2}\left[x-1\right] - \frac{1}{x^3}$

$= x^3 - 3x^2 + \cancel{3x} - 1 - \cancel{3x} + 6 - \cancel{\frac{3}{x}} + \cancel{\frac{3}{x}} - \frac{3}{x^2} - \frac{1}{x^3}$

$= x^3 - 3x^2 + 5 - \frac{3}{x^2} - \frac{1}{x^3}$ Ans.

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Q.6 Find the term involving:

i) x^4 in the expansion of $(3-2x)^7$

$a = 3, b = -2x, n = 7$

Consider

$T_{r+1} = \binom{n}{r} a^{n-r} b^r$

$T_{r+1} = \binom{7}{r}(3)^{7-r}(-2x)^r$

$= \binom{7}{r} 3^{7-r} \cdot (-2)^r x^r$

Put $r = 4$ to get x^4

$T_{4+1} = \binom{7}{4} 3^{7-4} (-2)^4 x^4$

$T_5 = (35)(27)(16)x^4$

$T_5 = 15120x^4$ Ans.

ii) x^{-2} in the expansion of $(x - \frac{2}{x^2})^{13}$

$a = x, b = \frac{-2}{x^2}, n = 13$

Consider $T_{r+1} = \binom{n}{r} a^{n-r} b^r$

$T_{r+1} = \binom{13}{r} x^{13-r} \left(\frac{-2}{x^2}\right)^r$

$= \binom{13}{r}(-2)^r \cdot x^{13-r} \cdot x^{-2r}$

$= \binom{13}{r}(-2)^r x^{13-3r}$

Put $13 - 3r = -2$ to get x^{-2}

$\Rightarrow 3r = 15 \Rightarrow r = 5$

$T_{5+1} = \binom{13}{5}(-2)^5 x^{-2}$

$T_6 = (1287)(-32)x^{-2}$

$T_6 = -41184x^{-2}$ Ans.

iii) a^4 in the expansion of $(\frac{2}{x} - a)^9$

$a = \frac{2}{x}, b = -a, n = 9$

Consider $T_{r+1} = \binom{n}{r} a^{n-r} b^r$

$T_{r+1} = \binom{9}{r}\left(\frac{2}{x}\right)^{9-r}(-a)^r$

$= \binom{9}{r}\left(\frac{2}{x}\right)^{9-r}(-1)^r a^r$

Put $r = 4$ to get a^4

$T_{4+1} = \binom{9}{4}\left(\frac{2}{x}\right)^{9-4}(-1)^4 a^4$

$T_5 = (126)\left(\frac{32}{x^5}\right) \cdot 1 \cdot a^4$

$T_5 = \frac{4032}{x^5}a^4$ Ans.

iv) y^3 in the expansion of $(x - \sqrt{y})^{11}$

$a = x, b = -\sqrt{y}, n = 11$

Consider $T_{r+1} = \binom{n}{r} a^{n-r} b^r$

$T_{r+1} = \binom{11}{r} x^{11-r}(-\sqrt{y})^r$

$= \binom{11}{r} x^{11-r} \cdot (-1)^r y^{r/2}$

Put $\frac{r}{2} = 3$ to get y^3

$\Rightarrow r = 6$

$T_{6+1} = \binom{11}{6} x^{11-6}(-1)^6 y^3$

$T_7 = 462x^5 \cdot 1 \cdot y^3$

$T_7 = 462x^5y^3$

Q.7 Find the Coeffient of:

i) x^5 in the expansion of $\left(x^2-\frac{3}{2x}\right)^{10}$

$a=x^2,\ b=\frac{-3}{2x},\ n=10$

Consider $T_{r+1}=\binom{n}{r}a^{n-r}b^r$

$$T_{r+1}=\binom{10}{r}(x^2)^{10-r}\left(\frac{-3}{2x}\right)^r$$
$$=\binom{10}{r}x^{20-2r}\cdot\frac{(-3)^r}{2^r x^r}$$
$$=\binom{10}{r}x^{20-3r}\cdot\frac{(-3)^r}{2^r}$$

Put $20-3r=5$ to get x^5

$\Rightarrow 3r=15 \Rightarrow r=5$

$$T_{5+1}=\binom{10}{5}\frac{(-3)^5}{2^5}x^5$$
$$T_6=252\cdot-\frac{243}{32}x^5$$
$$T_6=-\frac{15309}{8}x^5$$

Coefficient of $x^5=-\frac{15309}{8}$ Ans.

ii) x^n in the expansion of $\left(x^2-\frac{1}{x}\right)^{2n}$

$a=x^2,\ b=\frac{-1}{x},\ n=2n$

Consider $T_{r+1}=\binom{n}{r}a^{n-r}b^r$

$$T_{r+1}=\binom{2n}{r}(x^2)^{2n-r}\cdot\left(\frac{-1}{x}\right)^r$$
$$=\binom{2n}{r}x^{4n-2r}\cdot\frac{(-1)^r}{x^r}$$
$$=\binom{2n}{r}(-1)^r x^{4n-3r}$$

Put $4n-3r=n$ to get x^n

$3r=3n \Rightarrow n=r$

$$T_{n+1}=\binom{2n}{n}(-1)^n x^n$$
$$=\frac{(-1)^n \lfloor 2n}{\lfloor 2n-n \lfloor n}x^n$$
$$=\frac{(-1)^n\lfloor 2n}{(\lfloor n)^2}x^n$$

Coefficient of $x^n=\frac{(-1)^n\lfloor 2n}{(\lfloor n)^2}$ Ans.

Q.8 Find the 6th term of $\left(x^2-\frac{3}{2x}\right)^{10}$

$a=x^2,\ b=\frac{-3}{2x},\ n=10$

Consider $T_{r+1}=\binom{n}{r}a^{n-r}b^r$

$$T_6=\binom{10}{5}(x^2)^{10-5}\left(\frac{-3}{2x}\right)^5 \quad \text{for } r=5$$
$$T_6=(252)\,x^{10}\cdot\frac{-243}{32x^5}$$
$$T_6=-\frac{15309}{8}x^5 \quad \text{Ans.}$$

Q.9 Find the term independent of x:

i) $\left(x-\frac{2}{x}\right)^{10}$

$a=x,\ b=\frac{-2}{x},\ n=10$

Consider $T_{r+1}=\binom{n}{r}a^{n-r}b^r$

$$T_{r+1}=\binom{10}{r}x^{10-r}\left(\frac{-2}{x}\right)^r$$
$$=\binom{10}{r}(-2)^r x^{10-2r}$$

Put $10-2r=0$ to get x^0 term

$\Rightarrow r=5$

$$T_{5+1}=\binom{10}{5}(-2)^5x^0$$
$$T_6=252(-32)$$
$$T_6=-8064 \quad \text{Ans.}$$

ii) $\left(\sqrt{x}+\frac{1}{2x^2}\right)^{10}$

$a=\sqrt{x},\ b=\frac{1}{2x^2},\ n=10$

Consider $T_{r+1}=\binom{n}{r}a^{n-r}b^r$

$$T_{r+1}=\binom{10}{r}(\sqrt{x})^{10-r}\left(\frac{1}{2x^2}\right)^r$$
$$=\binom{10}{r}x^{5-\frac{r}{2}}\cdot\frac{1}{2^r x^{2r}}$$
$$=\binom{10}{r}\cdot\frac{1}{2^r}x^{5-\frac{5}{2}r}$$

Put $5-\frac{5r}{2}=0$ to get x^0 term

$\Rightarrow r=2$

$$T_{2+1}=\binom{10}{2}\cdot\frac{1}{2^2}x^0$$
$$T_3=\frac{45}{4} \quad \text{Ans.}$$

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iii) $(1+x^2)^3\left(1+\frac{1}{x^2}\right)^4$

$=(1+x^2)^3\left(\frac{1+x^2}{x^2}\right)^4$

$=\frac{(1+x^2)^7}{x^8}=\left[\frac{1+x^2}{x^{8/7}}\right]^7$

$=\left[x^{2-8/7}+\frac{1}{x^{8/7}}\right]^7$

$=\left[x^{6/7}+\frac{1}{x^{8/7}}\right]^7$

$a=x^{6/7}$, $b=\frac{1}{x^{8/7}}$, $n=7$

Consider $T_{r+1}=\binom{n}{r}a^{n-r}b^r$

$T_{r+1}=\binom{7}{r}(x^{6/7})^{7-r}\left(\frac{1}{x^{8/7}}\right)^r$

$=\binom{7}{r}x^{6-\frac{6r}{7}}\cdot\frac{1}{x^{\frac{8}{7}r}}$

$=\binom{7}{r}x^{6-2r}$

Put $6-2r=0$ to get x^0 term

$\Rightarrow r=3$

$T_{3+1}=\binom{7}{3}x^0$

$T_4=35$ Ans.

Q.10 Find the middle term of:

i) $\left(\frac{1}{x}-\frac{x^2}{2}\right)^{12}$

$a=\frac{1}{x}$, $b=-\frac{x^2}{2}$, $n=12$ (even)

so middle term $=\frac{n+2}{2}=\frac{12+2}{2}=\frac{14}{2}=7$th

Consider $T_{r+1}=\binom{n}{r}a^{n-r}b^r$ for $r=6$

$T_{6+1}=\binom{12}{6}\left(\frac{1}{x}\right)^{12-6}\left(-\frac{x^2}{2}\right)^6$

$T_7=(924)\,\frac{1}{x^6}\cdot\frac{x^{12}}{64}$

$T_7=\frac{231}{16}x^6$

which is required middle term.

---

ii) $\left(\frac{3}{2}x-\frac{1}{3x}\right)^{11}$

$a=\frac{3x}{2}$, $b=\frac{-1}{3x}$, $n=11$ (odd)

so middle terms are $\frac{n+1}{2}$ and $\frac{n+3}{2}$

$\frac{n+1}{2}=\frac{11+1}{2}=\frac{12}{2}=6$th

$\frac{n+3}{2}=\frac{11+3}{2}=\frac{14}{2}=7$th

Consider $T_{r+1}=\binom{n}{r}a^{n-r}b^r$

Let $r=5$

$T_{5+1}=\binom{11}{5}\left(\frac{3}{2}x\right)^{11-5}\left(\frac{-1}{3x}\right)^5$

$T_6=462\,\frac{3^6x^6}{2^6}\cdot\frac{-1}{3^5x^5}$

$=-\frac{462\cdot 3\cdot x}{64}$

$T_6=-\frac{693}{32}x$ Ans

Now let $r=6$

$T_{6+1}=\binom{11}{6}\left(\frac{3x}{2}\right)^{11-6}\left(\frac{-1}{3x}\right)^6$

$T_7=462\cdot\frac{3^5x^5}{2^5}\cdot\frac{1}{3^6x^6}$

$=\frac{462}{32\cdot 3x}=\frac{77}{16x}$ Ans.

---

iii) $\left(2x-\frac{1}{2x}\right)^{2m+1}$

$a=2x$, $b=\frac{-1}{2x}$, $n=2m+1$ (odd)

So middle terms are:

$\frac{n+1}{2}=\frac{2m+2}{2}=(m+1)$th

$\frac{n+3}{2}=\frac{2m+4}{2}=(m+2)$th

Consider $T_{r+1}=\binom{n}{r}a^{n-r}b^r$

Let $r=m$

$T_{m+1}=\binom{2m+1}{m}(2x)^{2m+1-m}\left(-\frac{1}{2x}\right)^m$

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$$T_{m+1} = \frac{(2m+1)!}{(2m+1-m)!\,m!}(2x)^{m+1} \cdot \frac{(-1)^m}{(2x)^m}$$

$$T_{m+1} = \frac{(2m+1)!}{(m+1)!\,m!}(-1)^m (2x) \quad \text{Ans.}$$

Now $r = m+1$

$$T_{m+2} = \binom{2m+1}{m+1}(2x)^{2m-m+1-1} \frac{(-1)^{m+1}}{(2x)^{m+1}}$$

$$T_{m+2} = \frac{(2m+1)!}{(2m+1-m-1)!\,(m+1)!}(2x)^m \cdot \frac{(-1)^{m+1}}{(2x)^{m+1}}$$

$$T_{m+2} = \frac{(2m+1)!}{m!\,(m+1)!} \frac{(-1)^{m+1}}{2x} \quad \text{Ans}$$

Q.11 Find $(2n+1)$ term from the end in the expansion of $\left(x-\frac{1}{2x}\right)^{3n}$.

From end $\left(\frac{-1}{2x}+x\right)^{3n}$

$a = \frac{-1}{2x}$, $b = x$, $n = 3n$

Consider $T_{r+1} = \binom{n}{r} a^{n-r} b^r$

let $r = 2n$

$$T_{2n+1} = \binom{3n}{2n}\left(\frac{-1}{2x}\right)^{3n-2n}(x)^{2n}$$

$$= \frac{(3n)!}{(3n-2n)!\,(2n)!} \frac{(-1)^n}{(2x)^n} \cdot x^{2n}$$

$$= \frac{(-1)^n (3n)!}{n!\,(2n)!} \cdot \frac{x^{2n}}{2^n \cdot x^n}$$

$$= \frac{(-1)^n (3n)!}{n!\,(2n)!} \frac{x^n}{2^n}$$

$$T_{2n+1} = \frac{(-1)^n (3n)!}{2^n\, n!\,(2n)!} x^n \quad \text{Ans.}$$

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Q.12 Show that middle term of $(1+x)^{2n}$ is $\frac{1.3.5\cdots(2n-1)}{n!}2^n x^n$.

Sol:- $a = 1$, $b = x$, $n = 2n$ (Even)

So middle term $= \frac{n+2}{2} = \frac{2n+2}{2} = (n+1)$th

Consider $T_{r+1} = \binom{n}{r} a^{n-r} b^r$

let $r = n$

$$T_{n+1} = \binom{2n}{n}(1)^{2n-n}(x)^n$$

$$= \frac{(2n)!}{(2n-n)!\,n!} \cdot (+1)^n x^n$$

$$= \frac{2n(2n-1)(2n-2)\ldots 4.3.2.1}{n!\,n!} x^n$$

$$= \frac{[2n(2n-2)\ldots 4.2][1.3.5\cdots(2n-1)]}{(n!)^2} x^n$$

$$= \frac{2^n.[n(n-1)\ldots 2.1][1.3.5\cdots(2n-1)]\,x^n}{(n!)^2}$$

$$= \frac{n!\; 1.3.5\cdots(2n-1)}{(n!)^2} 2^n . x^n$$

$$T_{n+1} = \frac{1.3.5.\ldots(2n-1)}{n!} 2^n x^n \quad \text{(Proved)}$$

Q.13 Show $\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\cdots+\binom{n}{n-1} = 2^{n-1}$

Sol:- We know that

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots$$
$$\cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$$

Put $a = 1$ $b = 1$

$$(1+1)^2 = \binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n-1}+\binom{n}{n}$$

$$2^n = \binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\binom{n}{3}+\cdots+\binom{n}{n-1}+\binom{n}{n} \quad \text{—— (1)}$$

Now put $a = 1$ $b = -1$

$$(1-1)^n = \binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\binom{n}{3}+\cdots-\binom{n}{n-1}+\binom{n}{n}$$

$$\binom{n}{0}+\binom{n}{2}+\cdots+\binom{n}{n} = \binom{n}{1}+\binom{n}{3}+\cdots+\binom{n}{n-1} \quad \text{—— (2)}$$

$\because (1-1)^n = 0$

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From ① and ②, we have

$$\left[\binom{n}{0}+\binom{n}{2}+\cdots+\binom{n}{n}\right]+\left[\binom{n}{1}+\binom{n}{3}+\cdots+\binom{n}{n-1}\right]=2^n$$

$$\left[\binom{n}{1}+\binom{n}{3}+\cdots+\binom{n}{n-1}\right]+\left[\binom{n}{1}+\binom{n}{3}+\cdots+\binom{n}{n-1}\right]=2^n \quad \text{by ②}$$

$$2\left[\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\cdots+\binom{n}{n-1}\right]=2^n$$

$$\Rightarrow \binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\cdots+\binom{n}{n-1}=2^{n-1} \quad \text{(Proved)}$$

Q.14 Show that $\binom{n}{0}+\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}+\cdots+\frac{1}{n+1}\binom{n}{n}=\frac{2^{n+1}-1}{n+1}$

$$\text{LHS}=\binom{n}{0}+\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}+\frac{1}{4}\binom{n}{3}+\cdots+\frac{1}{n+1}\binom{n}{n} \quad \because \binom{n}{0}=\binom{n}{n}=1$$

$$=1+\frac{1}{2}(n)+\frac{1}{3}\left(\frac{n(n-1)}{\underline{|2}}\right)+\frac{1}{4}\left[\frac{n(n-1)(n-2)}{\underline{|3}}\right]+\cdots+\frac{1}{n+1}(1)$$

$$=\frac{1}{n+1}\left[(n+1)+\frac{n(n+1)}{\underline{|2}}+\frac{(n+1)n(n-1)}{\underline{|3}}+\cdots+\frac{n+1}{n+1}\right]$$

$$=\frac{1}{n+1}\left[{}^{n+1}C_1+{}^{n+1}C_2+{}^{n+1}C_3+\cdots+{}^{n+1}C_{n+1}\right]$$

$$=\frac{1}{n+1}\left[\left({}^{n+1}C_0+{}^{n+1}C_1+{}^{n+1}C_2+\cdots+{}^{n+1}C_{n+1}\right)-{}^{n+1}C_0\right]$$

$$=\frac{1}{n+1}\left[2^{n+1}-{}^{n+1}C_0\right] \quad \because \sum_{r=0}^{n+1}\binom{n+1}{r}=2^{n+1}$$

$$=\frac{1}{n+1}\left[2^{n+1}-1\right] \quad \text{and } {}^{n+1}C_0=1$$

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$$=\frac{2^{n+1}-1}{n+1}=\text{RHS}$$

So

$$\binom{n}{0}+\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}+\frac{1}{4}\binom{n}{3}+\cdots+\frac{1}{n+1}\binom{n}{n}=\frac{2^{n+1}-1}{n+1} \quad \text{(Proved)}$$

Binomial Series:-

If n is a negative or fractional real number then

$$(1+x)^n=1+nx+\frac{n(n-1)}{\underline{|2}}x^2+\frac{n(n-1)(n-2)}{\underline{|3}}x^3+\cdots$$

is called Binomial Series which valids if $|x|<1$.

General term is $T_{r+1}=\frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}x^r$ for $|x|<1$