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Mathematical Induction and Binomial Theorem.

Mathematical Induction:-

Mathematical induction is a method used to testify any statement or Proposition.

This method was 1stly discovered by Francesco. Mourlico and he applied this principle Firstly For Positive integers "n".

Principle of Mathematical Induction:

The principle is based upon the following conditions:

Let S(n) be a statement or Proposition for +ve integers then

i) S(1) is true (S(n) is true for n=1)

ii) S(K+1) is true whenever S(K) is true then S(n) is true for all +ve integers.

* The step S(K) is true is called inductive hypothesis and S(K+1) is called conclusion.
* This principle can also be applicable for −ve and non-zero integers called extended principle of Mathematical induction.

EXERCISE: 8.1

Using mathematical induction, show that following fomulea are true for all +ve integers:

Q.1 $1+5+9+\cdots+(4n-3)=n(2n-1)$.

Let $n=1$

$LHS = 1$

$RHS = 1(2(1)-1) = 1(2-1) = 1$

$\Rightarrow LHS = RHS$

Condition (i) is satisfied.

Suppose result holds for $n=K$

$1+5+9+\cdots+(4K-3)=K(2K-1)$ —①

Now suppose $n=K+1$

$1+5+9+\cdots+(4K-3)+(4K+1)=(K+1)[2K+1]$

$LHS = \overline{1+5+9+\cdots+(4K-3)}+(4K+1)$

$= K(2K-1)+(4K+1)$ by ①

$= 2K^2-K+4K+1$

$= 2K^2+3K+1$

$= 2K^2+2K+K+1$

$= 2K(K+1)+1(K+1)$

$= (K+1)(2K+1) = RHS.$

Condition (ii) also holds so S(n) is true for all +ve integers.

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Q.2 $1+3+5+\cdots+(2n-1) = n^2$

Let $n=1$

$LHS = 1$

$RHS = (1)^2 = 1$

$\Rightarrow LHS = RHS$

Condition (i) holds.

Suppose result holds for $n=K$

$1+3+5+\cdots+(2K-1) = K^2$ —— ①

Now suppose $n=K+1$

$1+3+5+\cdots+(2K-1)+(2K+1) = (K+1)^2$

$LHS = \overline{1+3+5+\cdots+(2K-1)}+(2K+1)$

$= K^2 + (2K+1)$ by ①

$= (K+1)^2 = RHS$

Condition (2) also holds so $S(n)$ is true for all +ve integers.

Q.3 $1+4+7+\cdots+(3n-2) = \frac{n(3n-1)}{2}$

Let $n=1$

$LHS = 1$

$RHS = \frac{1[3(1)-1]}{2} = \frac{1.2}{2} = 1$

$\Rightarrow LHS = RHS$

Condition (i) holds

Suppose result holds for $n=K$

$1+4+7+\cdots+(3K-2) = \frac{K(3K-1)}{2}$ —— ①

Suppose $n=K+1$

$1+4+7+\cdots+(3K-2)+(3K+1) = \frac{(K+1)[3(K+1)-1]}{2}$

$= \frac{(K+1)(3K+2)}{2}$

$LHS = \overline{1+4+7+\cdots+(3K-2)}+(3K+1)$

$= \frac{K(3K-1)}{2} + 3K+1$ by ①

$= \frac{3K^2-K+2(3K+1)}{2}$

$= \frac{3K^2-K+6K+2}{2}$

$= \frac{3K^2+5K+2}{2}$

$= \frac{3K^2+3K+2K+2}{2}$

$= \frac{3K(K+1)+2(K+1)}{2}$

$= \frac{(K+1)(3K+2)}{2} = RHS$

Condition (ii) holds so $S(n)$ is true for all +ve integers.

Q.4 $1+2+4+\cdots+2^{n-1} = 2^n-1$

Let $n=1$

$LHS = 1$

$RHS = 2^1-1 = 2-1 = 1$

$\Rightarrow LHS = RHS$

Condition (i) holds.

Suppose result holds for $n=K$

$1+2+4+\cdots+2^{K-1} = 2^K-1$ —— ①

let $n=K+1$

$1+2+4+\cdots+2^{K-1}+2^K = 2^{K+1}-1$

$LHS = \overline{1+2+4+\cdots+2^{K-1}}+2^K$

$= (2^K-1)+2^K$ by ①

$= 2^K+2^K-1$

$= 2.2^K-1$

$= 2^{K+1}-1 = RHS$

Condition (ii) holds so $S(n)$ is true for all +ve integers.

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Q.5 $1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^{n-1}}=2\left[1-\frac{1}{2^n}\right]$

Let $n=1$

LHS $=1$

RHS $=2\left[1-\frac{1}{2}\right]=2\left[\frac{2-1}{2}\right]=1$

$\Rightarrow$ LHS $=$ RHS

Condition (i) holds.

Suppose result holds for $n=K$

$1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^{K-1}}=2\left[1-\frac{1}{2^K}\right]$ —①

Let $n=K+1$

$\underline{1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^{K-1}}+\frac{1}{2^K}}=2\left[1-\frac{1}{2^{K+1}}\right]$

LHS $=1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^{K-1}}+\frac{1}{2^K}$

$=2\left[1-\frac{1}{2^K}\right]+\frac{1}{2^K}$ by ①

$=2-\frac{2}{2^K}+\frac{1}{2^K}$

$=2-\frac{1}{2^K}=2-\frac{2}{2^K\cdot 2}$

$=2\left[1-\frac{1}{2^{K+1}}\right]=$ RHS

Condition (ii) holds so $S(n)$ is true for all +ve integers.

Q.6 $2+4+6+\cdots+2n=n(n+1)$

Let $n=1$

LHS $=2$

RHS $=1(1+1)=1\times 2=2$

$\Rightarrow$ LHS $=$ RHS

Condition (i) holds.

Suppose result holds for $n=K$

$2+4+6+\cdots+2K=(K+1)K$ —①

Now let $n=K+1$

$\underline{2+4+6+\cdots+2K+(2K+2)}=(K+1)(K+2)$

LHS $=2+4+6+\cdots+2K+(2K+2)$

$=K(K+1)+(2K+2)$ by ①

$=K^2+K+2K+2$

$=K^2+2K+K+2$

$=K(K+2)+1(K+2)$

$=(K+1)(K+2)=$ RHS

Condition (ii) holds so $S(n)$ is true for all +ve integers.

Q.7 $2+6+18+\cdots+2\times 3^{n-1}=3^n-1$

Let $n=1$

LHS $=2$

RHS $=3^1-1=3-1=2$

$\Rightarrow$ LHS $=$ RHS

Condition (i) holds.

Suppose result holds for $n=K$

$2+6+18+\cdots+2\times 3^{K-1}=3^K-1$ —①

Now $n=K+1$

$\underline{2+6+18+\cdots+2\times 3^{K-1}+2\times 3^K}=3^{K+1}-1$

LHS $=2+6+18+\cdots+2\times 3^{K-1}+2\times 3^K$

$=(3^K-1)+2\times 3^K$ by ①

$=3^K+2\times 3^K-1$

$=(1+2)3^K-1$

$=3\cdot 3^K-1$

$=3^{K+1}-1=$ RHS

Condition (ii) holds so $S(n)$ is true for all +ve integers.

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Q.8 $1\times3+2\times5+3\times7+\dots+n(2n+1)=\dfrac{n(n+1)(4n+5)}{6}$

Let $n=1$

$LHS = 1\times3 = 3$

$RHS = \dfrac{1(1+1)(4(1)+5)}{6} = \dfrac{1\times2\times9}{6} = 3$

$\Rightarrow LHS = RHS$

Condition (i) holds

suppose result holds for $n=K$

$1\times3+2\times5+3\times7+\dots+K(2K+1)=\dfrac{K(K+1)(4K+5)}{6}$ —①

Now $n=K+1$

$1\times3+2\times5+\dots+K(2K+1)+(K+1)(2K+3)$

$=\dfrac{(K+1)(K+2)[4(K+1)+5]}{6}$

$=\dfrac{(K+1)(K+2)(4K+9)}{6}$

$LHS = \overline{1\times3+2\times5+\dots+K(2K+1)}+(K+1)(2K+3)$

$=\dfrac{K(K+1)(4K+5)}{6}+(K+1)(2K+3)$ by ①

$=(K+1)\left[\dfrac{K(4K+5)}{6}+(2K+3)\right]$

$=(K+1)\left[\dfrac{4K^2+5K+12K+18}{6}\right]$

$=(K+1)\left[\dfrac{4K^2+17K+18}{6}\right]$

$=(K+1)\left[\dfrac{4K^2+8K+9K+18}{6}\right]$

$=(K+1)\left[\dfrac{4K(K+2)+9(K+2)}{6}\right]$

$=\dfrac{(K+1)(K+2)(4K+9)}{6} = RHS$

Condition (ii) holds so $S(n)$ is true for all +ve integers.

Q.9 $1\times2+2\times3+\dots+n(n+1)=\dfrac{n(n+1)(n+2)}{3}$

Let $n=1$

$LHS = 1\times2 = 2$

$RHS = \dfrac{1(1+1)(1+2)}{3} = \dfrac{1\times2\times3}{3} = 2$

$\Rightarrow LHS = RHS$

Condition (i) holds

suppose result holds for $n=K$

$1\times2+2\times3+\dots+K(K+1)=\dfrac{K(K+1)(K+2)}{3}$ —①

Now $n=K+1$

$1\times2+2\times3+\dots+K(K+1)+(K+1)(K+2)$

$=\dfrac{(K+1)(K+2)(K+3)}{3}$

$LHS = \underline{1\times2+2\times3+\dots+(K)(K+1)}+(K+1)(K+2)$

$=\dfrac{K(K+1)(K+2)}{3}+(K+1)(K+2)$ by ①

$=(K+1)(K+2)\left[\dfrac{K}{3}+1\right]$

$=\dfrac{(K+1)(K+2)(K+3)}{3} = RHS$

Condition (ii) holds so $S(n)$ is true for all +ve integers.

Q.10 $1\times2+3\times4+\dots+2n[2n-1]=\dfrac{n(n+1)(4n-1)}{3}$

Let $n=1$

$LHS = 1\times2 = 2$

$RHS = \dfrac{1(1+1)(4-1)}{3} = \dfrac{1\times2\times3}{3} = 2$

$\Rightarrow LHS = RHS$

Condition (i) holds

suppose result holds for $n=K$

$1\times2+3\times4+\dots+2K(2K-1)=\dfrac{K(K+1)(4K-1)}{3}$ —①

Now $n=K+1$

$1\times2+3\times4+\dots+2K(2K-1)+(2K+2)(2K+1)$

$=\dfrac{(K+1)(K+2)(4K+3)}{3}$

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$$LHS = \underline{1\times2+3\times4+\cdots+2K(2K-1)}+(2K+2)(2K+1)$$

$$= \frac{K(K+1)(4K-1)}{3} + 2(K+1)(2K+1) \quad \text{by } ①$$

$$= (K+1)\left[\frac{(4K-1)K}{3} + 2(2K+1)\right]$$

$$= (K+1)\left[\frac{4K^2-K+12K+6}{3}\right]$$

$$= \frac{(K+1)[4K^2+11K+6]}{3}$$

$$= \frac{(K+1)(4K^2+8K+3K+6)}{3}$$

$$= \frac{(K+1)[4K(K+2)+3(K+2)]}{3}$$

$$= \frac{(K+1)(K+2)(4K+3)}{3} = RHS$$

Condition (ii) holds so S(n) is true for all +ve integers.

Q.11 $\frac{1}{1\times2}+\frac{1}{2\times3}+\cdots+\frac{1}{n(n+1)} = 1-\frac{1}{n+1}$

Let $n=1$

$LHS = \frac{1}{1\times2} = \frac{1}{2}$

$RHS = 1-\frac{1}{1+1} = 1-\frac{1}{2} = \frac{2-1}{2} = \frac{1}{2}$

$\Rightarrow LHS = RHS$

Condition (i) holds

Suppose result holds for $n=K$

$$\frac{1}{1\times2}+\frac{1}{2\times3}+\cdots+\frac{1}{K(K+1)} = 1-\frac{1}{K+1} \quad — ①$$

Now $n=K+1$

$$\frac{1}{1\times2}+\frac{1}{2\times3}+\cdots+\frac{1}{K(K+1)}+\frac{1}{(K+1)(K+2)} = 1-\frac{1}{K+2}$$

$$LHS = \frac{1}{1\times2}+\frac{1}{2\times3}+\cdots+\frac{1}{K(K+1)}+\frac{1}{(K+1)(K+2)}$$

$$= 1-\frac{1}{K+1}+\frac{1}{(K+1)(K+2)} \quad \text{by } ①$$

$$= 1-\left(\frac{1}{K+1}-\frac{1}{(K+1)(K+2)}\right)$$

$$= 1-\frac{1}{K+1}\left(1-\frac{1}{K+2}\right)$$

$$= 1-\frac{1}{K+1}\left(\frac{K+2-1}{K+2}\right)$$

$$= 1-\frac{1}{\cancel{K+1}}\cdot\frac{\cancel{K+1}}{K+2}$$

$$= 1-\frac{1}{K+2} = RHS$$

Condition (ii) holds so S(n) is true for all +ve integers.

Q.12 $\frac{1}{1\times3}+\frac{1}{3\times5}+\cdots+\frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$

Let $n=1$

$LHS = \frac{1}{1\times3} = \frac{1}{3}$

$RHS = \frac{1}{2(1)+1} = \frac{1}{2+1} = \frac{1}{3}$

$\Rightarrow LHS = RHS$

Condition (i) holds

Suppose result holds for $n=K$

$$\frac{1}{1\times3}+\frac{1}{3\times5}+\cdots+\frac{1}{(2K-1)(2K+1)} = \frac{K}{2K+1} \quad — ①$$

Now $n=K+1$

$$\frac{1}{1\times3}+\frac{1}{3\times5}+\cdots+\frac{1}{(2K-1)(2K+1)}+\frac{1}{(2K+1)(2K+3)} = \frac{K+1}{2K+3}$$

$$LHS = \frac{1}{1\times3}+\frac{1}{3\times5}+\cdots+\frac{1}{(2K-1)(2K+1)}+\frac{1}{(2K+1)(2K+3)}$$

$$= \frac{K}{2K+1}+\frac{1}{(2K+1)(2K+3)}$$

$$= \frac{1}{2K+1}\left[K+\frac{1}{2K+3}\right]$$

$$= \frac{1}{2K+1}\left[\frac{2K^2+3K+1}{2K+3}\right]$$

$$= \frac{1}{2K+1}\left[\frac{2K^2+2K+K+1}{2K+3}\right]$$

$$= \frac{1}{2K+1}\left[\frac{2K(K+1)+1(K+1)}{2K+3}\right]$$

$$= \frac{1}{\cancel{2K+1}}\left[\frac{(K+1)\cancel{(2K+1)}}{2K+3}\right] = \frac{K+1}{2K+3}$$

$$= RHS$$

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Condition (ii) holds so S(n) is true for all +ve integers.

Q.13 $\frac{1}{2\times5}+\frac{1}{5\times8}+\cdots+\frac{1}{(3n-1)(3n+2)}=\frac{n}{2(3n+2)}$

Let $n=1$

$LHS=\frac{1}{2\times5}=\frac{1}{10}$

$RHS=\frac{1}{2(3+2)}=\frac{1}{2\times5}=\frac{1}{10}$

$\Rightarrow LHS=RHS$

Condition (i) holds

Suppose result for $n=K$

$\frac{1}{2\times5}+\frac{1}{5\times8}+\cdots+\frac{1}{(3K-1)(3K+2)}=\frac{K}{2(3K+2)}$ —(1)

let $n=K+1$

$\frac{1}{2\times5}+\frac{1}{5\times8}+\cdots+\frac{1}{(3K-1)(3K+2)}+\frac{1}{(3K+2)(3K+5)}=\frac{K+1}{2(3K+5)}$

$LHS=\frac{1}{2\times5}+\frac{1}{5\times8}+\cdots+\frac{1}{(3K-1)(3K+2)}+\frac{1}{(3K+2)(3K+5)}$

$=\frac{K}{2(3K+2)}+\frac{1}{(3K+2)(3K+5)}$ by (1)

$=\frac{1}{3K+2}\left[\frac{K}{2}+\frac{1}{3K+5}\right]$

$=\frac{1}{3K+2}\left[\frac{3K^2+5K+2}{3K+5}\right]$

$=\frac{1}{3K+2}\left[\frac{3K^2+3K+2K+2}{3K+5}\right]$

$=\frac{1}{3K+2}\left[\frac{3K(K+1)+2(K+1)}{3K+5}\right]$

$=\frac{(K+1)\cancel{(3K+2)}}{\cancel{(3K+2)}(3K+5)}=\frac{K+1}{3K+5}=RHS$

Condition (ii) holds so S(n) is true for all +ve integers.

Q.14 $r+r^2+\dots+r^n=\frac{r(1-r^n)}{1-r}$

Let $n=1$

$LHS=r$

$RHS=\frac{r\cancel{(1-r^1)}}{\cancel{1-r}}=r$

$\Rightarrow LHS=RHS$

Condition (i) holds

Suppose result holds for $n=K$

$r+r^2+\dots+r^K=\frac{r(1-r^K)}{1-r}$ —(1)

Let $n=K+1$

$r+r^2+\dots+r^K+r^{K+1}=\frac{r(1-r^{K+1})}{1-r}$

$LHS=\underline{r+r^2+\dots+r^K}+r^{K+1}$

$=\frac{r(1-r^K)}{1-r}+r^{K+1}$

$=\frac{r-\cancel{r^{K+1}}+\cancel{r^{K+1}}-r^{K+2}}{1-r}$

$=\frac{r-r^{K+2}}{1-r}=\frac{r(1-r^{K+1})}{1-r}$

$=RHS$

Condition (ii) holds so S(n) is true for all +ve integer.

Q.15 $a+(a+d)+\dots+(a+(n-1)d)=\frac{n}{2}[2a+(n-1)d]$.

Let $n=1$

$LHS=a$

$RHS=\frac{1}{2}[2a+(1-1)d]=\frac{1}{2}[2a]=a$

$\Rightarrow LHS=RHS$

Condition (i) holds

Suppose results for $n=K$

$a+(a+d)+\dots+(a+(K-1)d)=\frac{K}{2}[2a+(K-1)d]$ —(1)

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Let $n=K+1$

$$a+(a+d)+\dots+(a+(K-1)d)+(a+Kd)=\frac{(K+1)}{2}[2a+Kd]$$

LHS: $\underline{a+(a+d)+\dots+(a+(K-1)d)}+(a+Kd)$

$$=\frac{K}{2}[2a+(K-1)d]+(a+Kd)$$

$$=\frac{2aK+K^2d-Kd+2a+2Kd}{2}$$

$$=\frac{2aK+K^2d+2a+Kd}{2}$$

$$=\frac{2aK+2a+K^2d+Kd}{2}$$

$$=\frac{2a(K+1)+Kd(K+1)}{2}$$

$$=\frac{(K+1)}{2}[2a+Kd]=\text{RHS}.$$

Condition (ii) holds so $S(n)$ is true for all +ve integers.

Q.16 $1\underline{|1}+2\underline{|2}+\dots+n\underline{|n}=\underline{|n+1}-1$

Let $n=1$

LHS $=1\underline{|1}=1(1)=1$

RHS $=\underline{|1+1}-1=\underline{|2}-1=2-1=1$

$\Rightarrow$ LHS = RHS

Condition (i) holds.

Suppose result holds for $n=K$

$$1\underline{|1}+2\underline{|2}+\dots+K\underline{|K}=\underline{|K+1}-1 \quad \text{——} ①$$

Now $n=K+1$

$$1\underline{|1}+2\underline{|2}+\dots+K\underline{|K}+(K+1)\underline{|K+1}=\underline{|K+2}-1$$

LHS $=1\underline{|1}+2\underline{|2}+\dots+K\underline{|K}+(K+1)\underline{|K+1}$

$$=(\underline{|K+1}-1)+(K+1)\underline{|K+1} \quad \text{by } ①$$

$$=\underline{|K+1}+(K+1)\underline{|K+1}-1$$

$$=(1+K+1)\underline{|K+1}-1$$

$$=(K+2)\underline{|K+1}-1$$

$$=\underline{|K+2}-1 \quad \because \underline{|n}=n\underline{|n-1}$$

$$=\text{RHS}$$

Condition (ii) holds so $S(n)$ is true for all +ve integers.

Q.17 $a_n=a_1+(n-1)d$

Let $n=1$

LHS $=a_1$

RHS $=a_1+(1-1)d=a_1$

$\Rightarrow$ LHS = RHS

Condition (i) holds.

Suppose result holds for $n=K$.

$$a_K=a_1+(K-1)d \quad \text{——} ①$$

Now $n=K+1$

$$a_{K+1}=a_1+Kd$$

LHS $=a_{K+1}$

$$=a_K+d$$

$$=a_1+(K-1)d+d \quad \text{by } ①$$

$$=a_1+Kd-d+d$$

$$=a_1+Kd=\text{RHS}$$

Condition (ii) holds so $S(n)$ is true for all +ve integers.

Q.18 $a_n=a_1r^{n-1}$

Let $n=1$

LHS $=a_1$

RHS $=a_1r^{1-1}=a_1r^0=a_1$

$\Rightarrow$ LHS = RHS

Condition (i) holds

Suppose result holds for $n=K$

$$a_K=a_1r^{K-1} \quad \text{——} ①$$

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Now $n = K+1$

$a_{K+1} = a_1 r^K$

$LHS = a_{K+1}$

$= a_K \cdot r$

$= a_1 r^{K-1} \cdot r$ by ①

$= a_1 r^K = RHS$

Condition (ii) hold so $S(n)$ is true for all +ve integers.

Q.19 $1^2+3^2+5^2+\cdots+(2n-1)^2 = \dfrac{n(4n^2-1)}{3}$

Let $n=1$

$LHS = 1^2 = 1$

$RHS = \dfrac{1(4(1)^2-1)}{3} = \dfrac{4-1}{3} = \dfrac{3}{3} = 1$

$\Rightarrow LHS = RHS$

Condition (i) holds

Suppose result holds for $n=K$

$1^2+3^2+5^2+\cdots+(2K-1)^2 = \dfrac{K(4K^2-1)}{3}$ — ①

Now $n=K+1$

$1^2+3^2+5^2+\cdots+(2K-1)^2+(2K+1)^2 = \dfrac{(K+1)[4(K+1)^2-1]}{3}$

$= \dfrac{(K+1)[4K^2+8K+3]}{3}$

$LHS = \underline{1^2+3^2+\cdots+(2K-1)^2} + (2K+1)^2$

$= \dfrac{K(4K^2-1)}{3} + (2K+1)^2$ by ①

$= \dfrac{K(2K+1)(2K-1)}{3} + (2K+1)^2$

$= (2K+1)\left[\dfrac{(2K-1)K}{3} + (2K+1)\right]$

$= (2K+1)\left[\dfrac{2K^2-K+6K+3}{3}\right]$

$= \dfrac{(2K+1)[2K^2+5K+3]}{3}$

$= \dfrac{(2K+1)[2K^2+2K+3K+3]}{3}$

$= \dfrac{(2K+1)[2K(K+1)+3(K+1)]}{3}$

$= \dfrac{(K+1)[(2K+1)(2K+3)]}{3}$

$= \dfrac{(K+1)[4K^2+6K+2K+3]}{3}$

$= \dfrac{(K+1)(4K^2+8K+3)}{3} = RHS$

Condition (ii) holds so $S(n)$ is true for all +ve integers.

Q.20 $\binom{3}{3}+\binom{4}{3}+\cdots+\binom{n+2}{3} = \binom{n+3}{4}$

Let $n=1$

$LHS = \binom{3}{3} = 1$ $\quad (\because {}^3C_3 = {}^4C_4 = 1)$

$RHS = \binom{1+3}{4} = \binom{4}{4} = 1$

$\Rightarrow LHS = RHS$

Condition (i) holds

Suppose result holds for $n=K$

$\binom{3}{3}+\binom{4}{3}+\cdots+\binom{K+2}{3} = \binom{K+3}{4}$ — ①

Now $n=K+1$

$\binom{3}{3}+\binom{4}{3}+\cdots+\binom{K+2}{3}+\binom{K+3}{3} = \binom{K+4}{4}$

$LHS = \binom{3}{3}+\binom{4}{3}+\cdots+\binom{K+2}{3}+\binom{K+3}{3}$

$= \binom{K+3}{4}+\binom{K+3}{3}$ by ①

$= \binom{K+4}{4}$ $\quad (\because {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r)$

$= RHS$

Condition (ii) holds so $S(n)$ is true for all +ve integers.

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Q.21 Prove by Mathematical induction

(i) n^2+n is divisible by 2.

Let $n=1$ then

$n^2+n = (1)^2+(1) = 1+1=2$

which is divisible by 2

Condition (i) holds

Suppose result holds for $n=K$

$\Rightarrow K^2+K$ is divisible by 2

$K^2+K = 2q$ — ① where "q" is some quotient.

Now $n=K+1$

$(K+1)^2+(K+1) = K^2+2K+1+K+1$

$= (K^2+K)+2K+2$ by ①

$= 2q+2K+2$

$= 2(q+K+1)$

$= 2Q$ where $Q=q+K+1$

Clearly $(K+1)^2+(K+1)$ is divisible by 2 so $S(n)$ is true for all +ve integers.

(ii) 5^n-2^n is divisible by 3.

Let $n=1$ then

$5^n-2^n = 5-2=3$

which is divisible by 3

Condition (i) holds

Suppose result holds for $n=K$

5^K-2^K is divisible by 3

$\Rightarrow 5^K-2^K=3q$ — ① for some Quotent "q"

Now $n=K+1$

$5^{K+1}-2^{K+1} = 5\cdot 5^K-2^K\cdot 2$

$= 5(3q+2^K)-2\cdot 2^K$ $\because 5^K=3q+2^K$ by ①

$= 15q+5\cdot 2^K-2\cdot 2^K$

$= 15q+3\cdot 2^K$

$= 3[5q+2^K]$

$= 3Q$ where $Q=5q+2^K$

Clearly $5^{K+1}-2^{K+1}$ is divisible by 3 so $S(n)$ is true for all +ve integers.

(iii) 5^n-1 is divisible by 4.

Let $n=1$ then

$5^n-1 = 5-1=4$

which is divisible by 4

Condition (i) holds.

Suppose result holds for $n=K$

5^K-1 is divisible by 4

$\Rightarrow 5^K-1=4q$ — ① for some qoutient "q"

Now $n=K+1$

$5^{K+1}-1 = 5\cdot 5^K-1$

$= 5(4q+1)-1$ $\because 5^K=4q+1$ by ①

$= 20q+5-1$

$= 20q+4$

$= 4[5q+1]$

$= 4Q$ where $Q=5q+1$

which is divisible by 4

so $S(n)$ is divisible by 4 for all +ve integer.

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iv) $8\times10^n-2$ is divisible by 6.

Let $n=1$

$8\times10^n-2 = 8\times10^1-2 = 78$

which is divisible by 6

Suppose result holds for $n=K$

$\Rightarrow 8\times10^K-2$ is divisible by 6.

$\Rightarrow 8\times10^K-2 = 6q$ where "q" is some quotient ——(1)

Now $n=K+1$

$8\times10^{K+1}-2 = 8\times10^K.10-2$

$=10(6q+2)-2$ $\quad\because 8\times10^K=6q+2$ by (1)

$=60q+20-2$

$=60q+18$

$=6[10q+3]$

$=6Q$ where $Q=10q+3$

which is divisible by 6 so $S(n)$ is true for all +ve integers.

(v) n^3-n is divisible by 6

Let $n=1$

$n^3-n = (1)^3-(1) = 0$

which is divisible by 6

Suppose result holds for $n=K$.

so K^3-K is divisible by 6

$\Rightarrow K^3-K=6q$ where "q" is some quotient ——(1)

Now $n=K+1$

$(K+1)^3-(K+1) = K^3+3K^2+3K+1-K-1$

$=(K^3-K)+(3K^2+3K)$

$=6q+3K(K+1)$ by (1)

$\because K(K+1)$ is always even so

$K(K+1)=2p$ for $p\in Z^+$

$=6q+3(2p)$

$=6q+6p$

$=6(q+p)$

$=6Q$ where $Q=q+p$

which is divisible by 6 so $S(n)$ is true for all +ve integers.

Q.22 $\frac{1}{3}+\frac{1}{3^2}+\dots+\frac{1}{3^n}=\frac{1}{2}\left[1-\frac{1}{3^n}\right]$

Let $n=1$

LHS $=1/3$

RHS $=\frac{1}{2}\left[1-\frac{1}{3^1}\right]=\frac{1}{2}\left[\frac{3-1}{3}\right]=\frac{1}{3}$

$\Rightarrow$ LHS = RHS

Condition (i) holds

Suppose result holds for $n=K$

$\frac{1}{3}+\frac{1}{3^2}+\dots+\frac{1}{3^K}=\frac{1}{2}\left[1-\frac{1}{3^K}\right]$ ——(1)

Now $n=K+1$

$\frac{1}{3}+\frac{1}{3^2}+\dots+\frac{1}{3^K}+\frac{1}{3^{K+1}}=\frac{1}{2}\left[1-\frac{1}{3^{K+1}}\right]$

LHS $=\frac{1}{3}+\frac{1}{3^2}+\dots+\frac{1}{3^K}+\frac{1}{3^{K+1}}$

$=\frac{1}{2}\left[1-\frac{1}{3^K}\right]+\frac{1}{3^{K+1}}$ by (1)

$=\frac{1}{2}\left[1-\frac{1}{3^K}\right]+\frac{2}{2.3^{K+1}}$

$=\frac{1}{2}\left[1-\frac{1}{3^K}+\frac{2}{3^{K+1}}\right]$

$=\frac{1}{2}\left[1-\left(\frac{1}{3^K}-\frac{2}{3^K.3}\right)\right]$

$=\frac{1}{2}\left[1-\frac{3-2}{3^K.3}\right]$

$=\frac{1}{2}\left[1-\frac{1}{3^{K+1}}\right]=$ RHS

Condition (ii) holds so $S(n)$ is true for all +ve integers.

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Q.23 $1^2-2^2+3^2-4^2+\dots+(-1)^{n-1}n^2=(-1)^{n-1}\dfrac{n(n+1)}{2}$

Let $n=1$

LHS $= 1^2 = 1$

RHS $= (-1)^{1-1}\dfrac{1(1+1)}{2} = (-1)^0\dfrac{2}{2} = 1$

$\Rightarrow$ LHS = RHS

Condition (i) holds

Suppose result holds for $n=K$

$1^2-2^2+3^2-4^2+\dots+(-1)^{K-1}K^2=(-1)^{K-1}\dfrac{K(K+1)}{2}$ —①

Now $n=K+1$

$1^2-2^2+\dots+(-1)^{K-1}K^2+(-1)^K(K+1)^2=(-1)^K\dfrac{(K+1)(K+2)}{2}$

LHS $= 1^2-2^2+\dots+(-1)^{K-1}K^2+(-1)^K(K+1)^2$

$= (-1)^{K-1}\dfrac{K(K+1)}{2}+(-1)^K(K+1)^2$

$=(-1)^K(K+1)\left[(-1)^{-1}\dfrac{K}{2}+K+1\right]$

$=(-1)^K(K+1)\left[K+1-\dfrac{K}{2}\right]$

$=(-1)^K(K+1)\left[\dfrac{2K+2-K}{2}\right]$

$=(-1)^K(K+1)\left[\dfrac{K+2}{2}\right]$ = RHS

Condition (ii) holds so $S(n)$ is true for all +ve integers.

Q.24 $1^3+3^3+5^3+\dots+(2n-1)^3=n^2[2n^2-1]$

Let $n=1$

LHS $= 1^3 = 1$

RHS $= 1^2[2(1)^2-1] = 1.(2-1) = 1$

$\Rightarrow$ LHS = RHS

Condition (i) holds

Suppose result holds for $n=K$

$1^3+3^3+\dots+(2K-1)^3=K^2[2K^2-1]$ —①

Now $n=K+1$

$1^3+3^3+\dots+(2K-1)^3+(2K+1)^3=(K+1)^2[2(K+1)^2-1]$

$=(K+1)^2[2(K^2+2K+1)-1]$

$=(K+1)^2[2K^2+4K+1]$

LHS $=1^3+3^3+5^3+\dots+(2K-1)^3+(2K+1)^3$

$= K^2[2K^2-1]+(2K+1)^3$

$=2K^4-K^2+8K^3+12K^2+6K+1$

$=2K^4+8K^3+11K^2+6K+1$

$=2K^4+2K^3+6K^3+6K^2+5K^2+5K+K+1$

$=2K^3(K+1)+6K^2(K+1)+5K(K+1)+(K+1)$

$=(K+1)[2K^3+6K^2+5K+1]$

$=(K+1)[2K^3+2K^2+4K^2+4K+K+1]$

$=(K+1)[2K^2(K+1)+4K(K+1)+(K+1)]$

$=(K+1)^2[2K^2+4K+1]$ = RHS

Condition (ii) holds so $S(n)$ is true for all +ve integers.

Q.25 $x+1$ is a factor of $x^{2n}-1$; $x\neq-1$

Let $n=1$

$x^{2n}-1 = x^2-1 = (x+1)(x-1)$

clearly $x+1$ is a factor of $S(n)$

Suppose result holds for $n=K$

$\Rightarrow x+1$ is a factor of $x^{2K}-1$.

$\Rightarrow x^{2K}-1=(x+1)q$ for some "q" quotient. —①

Now $n=K+1$

$x^{2K+2}-1=x^{2K}\cdot x^2-1$

$=x^{2K}\cdot x^2-x^2+x^2-1$

$=x^2(x^{2K}-1)+(x^2-1)$

$=x^2(x+1)q+(x-1)(x+1)$ by ①

$=(x+1)[x^2q+(x-1)]$

$=(x+1)Q$ where $Q=x^2q+x-1$

$\Rightarrow x+1$ is factor so $S(K+1)$ so true for all +ve integers.

Q.26 $x-y$ is a factor of $x^n - y^n$

Let $n=1$

$x^n - y^n = x^1 - y^1 = x-y$

Clearly $x-y$ is its factor.

Suppose result holds for $n=K$

$x-y$ is a factor of $x^K - y^K$

$\Rightarrow \quad x^K - y^K = (x-y)q$ for some "q" quotient. ——(1)

Now $n = K+1$

$$x^{K+1} - y^{K+1} = x^K \cdot x - xy^K + xy^K - yy^K$$
$$= x(x^K - y^K) + (x-y)y^K$$
$$= x(x-y)q + (x-y)y^K \quad \text{by (1)}$$
$$= (x-y)[xq + y^K]$$
$$= (x-y)Q$$

where $Q = xq + y^K$

Clearly $x-y$ is a factor of $S(K+1)$

so $S(n)$ is true for all +ve integers.

Q.27 $x+y$ is a factor of $x^{2n-1} + y^{2n-1}$

Let $n=1$

$x^{2n-1} + y^{2n-1} = x^{2-1} + y^{2-1} = x+y$

$\Rightarrow x+y$ is factor of $S(1)$

Now suppose result holds for $n=K$

$\Rightarrow x+y$ is a factor of $x^{2K-1} + y^{2K-1}$

$\Rightarrow \quad x^{2K-1} + y^{2K-1} = (x+y)q$ where "q" is some quotient. ——(1)

Now $n = K+1$

$$x^{2K+1} + y^{2K+1} = x^2 \cdot x^{2K-1} + y^2 \cdot y^{2K-1}$$
$$= x^2 x^{2K-1} + x^2 y^{2K-1} - x^2 y^{2K-1} + y^2 y^{2K-1}$$
$$= x^2[x^{2K-1} + y^{2K-1}] - y^{2K-1}[x^2 - y^2]$$
$$= x^2(x+y)q - y^{2K-1}(x+y)(x-y) \quad \text{by (1)}$$
$$= (x+y)[x^2 q - y^{2K-1}(x-y)]$$
$$= (x+y)Q \quad \text{for } Q = x^2 q - y^{2K-1}(x-y)$$

which shows $x+y$ is its factor

so $S(n)$ is true for all +ve integers.

Q.28 $1 + 2 + 2^2 + \dots + 2^n = 2^{n+1} - 1$

Let $n=0$ for non-negative

LHS $= 1$

RHS $= 2^{0+1} - 1 = 2 - 1 = 1$

$\Rightarrow$ LHS = RHS

Condition (i) holds

Suppose result holds for $n=K$

$1 + 2 + 2^2 + \dots + 2^K = 2^{K+1} - 1$ ——(1)

Now $n = K+1$

$1 + 2 + 2^2 + \dots + 2^K + 2^{K+1} = 2^{K+2} - 1$

$$\text{LHS} = 1 + 2 + 2^2 + \dots + 2^K + 2^{K+1}$$
$$= (2^{K+1} - 1) + 2^{K+1} \quad \text{by (1)}$$
$$= 2^{K+1} + 2^{K+1} - 1$$
$$= 2 \cdot 2^{K+1} - 1$$
$$= 2^{K+2} - 1$$
$$= \text{RHS}$$

Condition (ii) holds so $S(n)$ is true for all +ve integers.

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Q.29 Given that $AB = BA$

Let $n=1$

$LHS = AB^1 = AB$

$RHS = B^1A = BA = AB \quad \because AB = BA$

$\Rightarrow LHS = RHS$

Condition (i) holds.

Now suppose result holds for $n=K$

$$AB^K = B^KA \quad \text{—①}$$

Now $n = K+1$

$$AB^{K+1} = B^{K+1}A$$

$LHS = AB^{K+1}$

$= (AB^K)B$

$= (B^KA)B \quad$ by ①

$= B^K(AB)$

$= B^K(BA)$

$= (B^K.B)A$

$= B^{K+1}.A = RHS$

Condition (ii) holds so $S(n)$ is true for all +ve integers.

Q.30 n^2-1 is divisible by 8 where n being odd +ve integer.

Let $n=1$

$n^2-1 = (1)^2-1 = 0$

which is divisible by 8

Now suppose result holds for $n=K$

K^2-1 is divisible by 8 $\quad$ K being odd

$\Rightarrow K^2-1 = 8q$ —① "q" being quotient

Now $n = K+2$ (odd)

$(K+2)^2 - 1 = K^2+4K+4 \;\; -1$

$= K^2+4K+4-1$

$= (K^2-1) + 4(K+1)$

$= 8q + 4(2p) \quad$ by ①

where $K+1$ is even so $K+1 = 2p$

$= 8q + 8p$

$= 8(q+p)$

$= 8Q \quad$ where $Q = q+p$

which is divisible by 8 so $S(n)$ is true for all odd +ve integers.

Q.31 $\ln x^n = n \ln x \; ; \; x>0, \; n \geq 0$

Let $n=1$

$LHS = \ln x^1 = \ln x$

$RHS = 1.\ln x = \ln x$

$\Rightarrow LHS = RHS$

Condition (i) holds

Suppose result holds for $n=K$

$$\ln x^K = K \ln x \quad \text{—①}$$

Now $n = K+1$

$$\ln x^{K+1} = (K+1)\ln x$$

$LHS = \ln x^{K+1}$

$= \ln(x^K.x)$

$= \ln x^K + \ln x \quad \because \ln mn = \ln m + \ln n$

$= K\ln x + \ln x \quad$ by ①

$= (K+1)\ln x$

$= RHS$

Condition (ii) holds so $S(n)$ is true for all +ve integers.

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Q.32 $n! > 2^n - 1 \quad \forall \; n \geq 4$

Let $n = 4$

$4! > 2^4 - 1$

$24 > 16 - 1 \Rightarrow 24 > 15$

which is true for $n = 4$

Suppose result holds for $n = K$

so $K! > 2^K - 1$ —①

Now $n = K+1$

$(K+1)! = (K+1)K! > (K+1)(2^K - 1)$ by①

$(K+1)! > K(2^K - 1) + 2^K - 1$

$(K+1)! > K \cdot 2^K - K + 2^K - 1$

$(K+1)! > (K+1)2^K - K - 1$

$\because (K+1)2^K - 1 > 2^{K+1} \quad \forall K \geq 4$

$\Rightarrow (K+1)! > 2^{K+1} - 1$

Condition (ii) holds so $S(n)$ is true for all +ve integers ≥ 4.

Q.33 $n^2 > n+3 \quad \forall \; n \geq 3$

Let $n = 3$

$3^2 > 3+3$

$9 > 6$ which is true for $n=3$

Suppose result holds for $n = K$

$K^2 > K + 3$ —①

For $n = K+1$

$(K+1)^2 = K^2 + 2K + 1$

$(K+1)^2 = K + (K^2 + K + 1)$

$\because K^2 + K + 1 > 4 \quad \forall K \geq 3$

$(K+1)^2 > K + 4$

Condition (ii) holds so $S(n)$ is true for all +ve integers ≥ 3.

Q.34 $4^n > 3^n + 2^{n-1} \quad \forall \; n \geq 2$

Let $n = 2$

$4^2 > 3^2 + 2^{2-1}$

$16 > 9 + 2$

$16 > 11$ which is true

Suppose result holds for $n = K$

$4^K > 3^K + 2^{K-1}$

Now for $n = K+1$, multiplying both sides by 4

$4 \cdot 4^K > 4(3^K + 2^{K-1})$

$4^{K+1} > 4 \cdot 3^K + 4 \cdot 2^{K-1}$

$4^{K+1} > 3 \cdot 3^K + 2 \cdot 2^{K-1}$

$\because 4 > 3 \;,\; 4 > 2$

$4^{K+1} > 3^{K+1} + 2^K$

Condition (ii) holds so $S(n)$ is true for all +ve integers ≥ 2.

Q.35 $3^n < n! \quad \forall \; n > 6$

Let $n = 7$

$3^7 < 7!$

$2187 < 5040$

which is true

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Suppose result for $n = K$

$3^K < K!$

Now for $n = K+1$, multiplying by 3, we have

$3 \cdot 3^K < 3K!$

$3^{K+1} < (K+1)(K!) \quad \because 3 < K+1$ as $K > 6$

$3^{K+1} < (K+1)!$

Condition (ii) holds so $S(n)$ is true for all +ve integers > 6.

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Q.36 $n! > n^2 \quad \forall n \geq 4$

Let $n = 4$

$4! > 4^2$

$24 > 16$ which is true

Suppose result holds for $n = K$

$K! > K^2$ — ①

Now for $n = K+1$, multiplying by $K+1$, we have

$(K+1)K! > (K+1)K^2$

$(K+1)! > (K+1)(K+1) \quad \because K \geq 4$ so $K+1 < K^2$

$(K+1)! > (K+1)^2$

Condition (ii) holds so $S(n)$ is true for all +ve integers ≥ 4.

Q.37 $3+5+7+\dots+(2n+5) = (n+2)(n+4) \quad \forall n \geq -1$

Let $n = -1$

LHS = 3

RHS = $[-1+2][-1+4] = 1 \times 3 = 3$

$\Rightarrow$ LHS = RHS

Condition (i) holds

Let result holds for $n = K$

$3+5+7+\dots+(2K+5) = (K+2)(K+4)$ — ①

Now for $n = K+1$

$3+5+7+\dots+(2K+5)+(2K+7) = (K+3)(K+5)$

LHS = $\overline{3+5+7+\dots+(2K+5)} + (2K+7)$

$= (K+2)(K+4) + (2K+7)$ by ①

$= K^2 + 4K + 2K + 8 + 2K + 7$

$= K^2 + 8K + 15$

$= K^2 + 3K + 5K + 15$

$= K(K+3) + 5(K+3)$

$= (K+3)(K+5) =$ RHS.

Condition (ii) holds so $S(n)$ is true for all integral values of $n \geq -1$

Q.38 $1 + nx < (1+x)^n$

where $x > -1$ and $n \geq 2$

Let $n = 2$

$1 + 2x < (1+x)^2$

$1 + 2x < 1 + 2x + x^2$

which is true for $n = 2$

Suppose result holds for $n = K$

$\Rightarrow \quad 1 + Kx < (1+x)^K$

For $n = K+1$, multiplying both sides by $1+x$

$(1+Kx)(1+x) < (1+x)^K \cdot (1+x)$

$1 + Kx + x + Kx^2 < (1+x)^{K+1}$

$1 + (K+1)x + Kx^2 < (1+x)^{K+1}$

$1 + (K+1)x < (1+x)^{K+1}$

Condition (ii) holds so $S(n)$ is true for all integral values of $n \geq 2$ and for $x > -1$.

Binomial Expression:-

"An algebraic expression containing two terms is called binomial expression."

e.g $(a+x), (a+x)^2, (a+x)^3, \dots$ are binomial expressions.