$$\text{Product}(P) = \left(\frac{1-\alpha}{1+\alpha}\right)\left(\frac{1-\beta}{1+\beta}\right)$$

$$P = \frac{1-\alpha-\beta+\alpha\beta}{1+\alpha+\beta+\alpha\beta}$$

$$P = \frac{1-(\alpha+\beta)+\alpha\beta}{1+(\alpha+\beta)+\alpha\beta}$$

$$P = \frac{1-3+5}{1+3+5} = \frac{3}{9} = \frac{1}{3}$$

Thus Equation is $y^2 - Sy + P = 0$

$$y^2 + \frac{8}{9}y + \frac{1}{3} = 0$$

$$9y^2 + 8y + 3 = 0 \text{ (Req)}.$$

Nature of roots:-

The solution of $ax^2+bx+c=0$ is $x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$.

The part $\pm\sqrt{b^2-4ac}$ gives the possible values of x.

b^2-4ac is called discriminant of Eq.

These are the cases:-

* If $b^2-4ac > 0$ then roots will be real and distinct (unequal)
* If $b^2-4ac = 0$ then roots will be real and Equal.
* If $b^2-4ac < 0$ then roots will be imaginary
* If b^2-4ac is a Perfect Square then roots will be rational and distinct.

Exercise: 4.7

Q.1 Discuss the nature

(i) $4x^2+6x+1=0$

$a=4 \quad b=6 \quad c=1$

$b^2-4ac = (6)^2-4(4)(1)$

$= 36-16 = 20 > 0$

Roots are real and distinct.

(ii) $x^2-5x+6=0$

$a=1 \quad b=-5 \quad c=6$

$b^2-4ac = (-5)^2-4(1)(6)$

$= 25-24 = 1 > 0$

Roots are real and distinct.

(iii) $2x^2-5x+1=0$

$a=2 \quad b=-5 \quad c=1$

$b^2-4ac = (-5)^2-4(2)(1)$

$= 25-8 = 17 > 0$

Roots are real and distinct.

(iv) $25x^2-30x+9=0$

$a=25 \quad b=-30 \quad c=9$

$b^2-4ac = (30)^2-4(25)(9)$

$= 900-900 = 0$

Roots are real and Equal.

Q.2 Show that roots are real for:

(i) $x^2 - 2(m+1/m)x + 4 = 0$

$a=1$, $b=-2(m+1/m)$, $c=4$

$b^2-4ac = 4(m+1/m)^2 - 4(1)(4)$

$= 4\{m+1/m\}^2 - 16$

$= 4(m^2+\frac{1}{m^2}+2) - 16$

$= 4m^2+\frac{4}{m^2}+8-16$

$= 4m^2+\frac{4}{m^2}-8$

$= 4(m^2+1/m^2-2)$

$= [2\{m-1/m\}]^2 > 0$

Thus roots are real

(ii) $(b-c)x^2+(c-a)x+(a-b)=0$ $a,b,c \in Q$

$A=(b-c)$ $B=(c-a)$ $C=(a-b)$

$B^2-4AC = (c-a)^2 - 4(b-c)(a-b)$

$= (c^2+a^2-2ac) - 4(ab-b^2-ac+bc)$

$= c^2+a^2-2ac-4ab+4b^2+4ac-4bc$

$= c^2+a^2+4b^2+2ac-4ab-4bc$

$= a^2+4b^2+c^2-2(a)(2b)-2(2b)(c) + 2(c)(a)$

$= (a-2b+c)^2 > 0$

Thus roots are real.

Q.3 Show that roots are rational for:

(i) $(p+q)x^2 - px - q = 0$

$a=(p+q)$ $b=-p$ $c=-q$

$b^2-4ac = (-p)^2 - 4(p+q)(-q)$

$= p^2 + 4(pq+q^2)$

$= p^2+4pq+q^2$

TAHIR

$= p^2+4q^2+4pq$

$= (p+2q)^2$

$\because$ b^2-4ac is perfect square

So roots are rational

* If b^2-4ac is not a perfect square then roots are irrational.

(ii) $px^2-(p-q)x-q=0$

$a=p$ $b=-(p-q)$ $c=-q$

$b^2-4ac = [-(p-q)]^2 - 4(p)(-q)$

$= p^2+q^2-2pq+4pq$

$= p^2+q^2+2pq$

$= (p+q)^2$

Thus roots are rational.

Q.4 For what value of m, roots are equal:

(i) $(m+1)x^2+2(m+3)x+(m+8)=0$

$a=m+1$ $b=2(m+3)$ $c=m+8$

The roots will be equal if

$b^2-4ac=0$

$\Rightarrow [2(m+3)]^2 - 4[(m+1)(m+8)] = 0$

$\Rightarrow 4(m^2+9+6m) - 4(m^2+8m+m+8) = 0$

$\Rightarrow \cancel{4m^2}+36+24m-\cancel{4m^2}-32m-4m-32=0$

$\Rightarrow -12m+4=0$

$12m=4 \Rightarrow m=4/12$

$\boxed{m=1/3}$

roots will be equal if $m=1/3$

(ii) $x^2-2(1+3m)x+7(3+2m)=0$

$a=1 \quad b=-2(1+3m) \quad c=7(3+2m)$

$b^2-4ac=0$ for equal roots.

$\Rightarrow [-2(1+3m)]^2-4(1)[7(3+2m)]=0$

$\Rightarrow 4(1+9m^2+6m)-28(3+2m)=0$

$\Rightarrow 36m^2+4+24m-84-56m=0$

$36m^2-32m-80=0$

$4\{9m^2-8m-20\}=0 \quad 4\neq 0$

$9m^2-8m-20=0$

$9m^2-18m+10m-20=0$

$(9m+10)(m-2)=0$

$m-2=0 \quad \wedge \quad 9m+10=0$

$m=2 \quad \wedge \quad m=-10/9$

Thus roots will be Equal if

$m=2$ or $m=-10/9$.

(iii) $(1+m)x^2-2(1+3m)x+(1+8m)=0$

$a=1+m \quad b=-2(1+3m) \quad c=1+8m$

$b^2-4ac=0$ for equal roots

$\Rightarrow [-2(1+3m)]^2-4[(1+m)(1+8m)]=0$

$\Rightarrow 4(1+9m^2+6m)-4(1+m+8m+8m^2)=0$

$\Rightarrow \cancel{4}+36m^2+24m-\cancel{4}-4m-32m-32m^2=0$

$4m^2-12m=0$

$4m(m-3)=0 \quad 4\neq 0$

$m(m-3)=0$

$m=0 \quad \wedge \quad m-3=0$

$m=3$

Root will be equal if $m=0, 3$

Q.5 $x^2+(mx+c)^2=a^2$

$x^2+(m^2x^2+c^2+2mxc)-a^2=0$

$(m^2+1)x^2+(2mc)x+(c^2-a^2)=0$

$a=m^2+1 \quad b=2mc \quad c=c^2-a^2$

$b^2-4ac=0$ for equal roots

$\Rightarrow (2mc)^2-4(m^2+1)(c^2-a^2)=0$

$\Rightarrow 4m^2c^2-4(m^2c^2+c^2-m^2a^2-a^2)=0$

$\Rightarrow \cancel{4m^2c^2}-\cancel{4m^2c^2}-4c^2+4m^2a^2+4a^2=0$

$\Rightarrow 4m^2a^2+4a^2=4c^2$

$4a^2\{m^2+1\}=4c^2$

$a^2(1+m^2)=c^2$ (Proved)

Q.6 $(mx+c)^2=4ax$

$m^2x^2+c^2+2mcx-4ax=0$

$m^2x^2+(2mc-4a)x+c^2=0$

$a=m^2 \quad b=(2mc-4a) \quad c=c^2$

$b^2-4ac=0$ for equal roots

$\Rightarrow (2mc-4a)^2-4(m^2)(c^2)=0$

$\Rightarrow \cancel{4m^2c^2}+16a^2-16mca-\cancel{4m^2c^2}=0$

$\Rightarrow 16a^2-16mca=0$

$16a(a-mc)=0 \quad 16\neq 0, \; a\neq 0$

$a(a-mc)=0$

$a=mc=c$

$\Rightarrow c=a/m$ (Proved).

Q.7 $\frac{x^2}{a^2}+\frac{(mx+c)^2}{b^2}=1$

$$b^2x^2+a^2(m^2x^2+c^2+2mxc)=a^2b^2$$

$$x^2(b^2+a^2m^2)+a^2c^2+(2a^2mc)x-a^2b^2=0$$

$$(b^2+a^2m^2)x^2+(2a^2mc)x+a^2(c^2-b^2)=0$$

$A=b^2+a^2m^2 \quad B=2a^2mc \quad C=a^2(c^2-b^2)$

$B^2-4AC=0$ for Equal roots

$$\Rightarrow (2a^2mc)^2-4(b^2+a^2m^2)(a^2c^2-a^2b^2)=0$$

$$\Rightarrow 4a^4m^2c^2-4(a^2b^2c^2-a^2b^4+a^4m^2c^2-a^4m^2b^2)=0$$

$$\Rightarrow \cancel{4a^4m^2c^2}-4a^2b^2c^2+4a^2b^4-\cancel{4a^4m^2c^2}+4a^4m^2b^2=0$$

$$\Rightarrow 4a^4m^2b^2-4a^2b^2c^2+4a^2b^4=0$$

$$\Rightarrow 4a^2b^2(a^2m^2-c^2+b^2)=0$$

$$\Rightarrow a^2m^2-c^2+b^2=0 \qquad 4a^2b^2\neq 0$$

$$c^2=a^2m^2+b^2 \quad \text{(Proved)}.$$

Q.8 $(a^2-bc)x^2+2(b^2-ca)x+(c^2-ab)=0$

$A=a^2-bc \quad B=2(b^2-ca) \quad C=c^2-ab$

$B^2-4AC=0$ for equal roots.

$$\Rightarrow [2(b^2-ca)]^2-4[(a^2-bc)(c^2-ab)]=0$$

$$\Rightarrow 4(b^4+c^2a^2-2b^2ca)-4(a^2c^2-a^3b-bc^3+ab^2c)=0$$

$$4b^4+\cancel{4c^2a^2}-\underline{8b^2ca}-\cancel{4a^2c^2}+4a^3b+4bc^3-\underline{4ab^2c}=0$$

$$\Rightarrow 4b^4-12ab^2c+4a^3b+4bc^3=0$$

$$4b(b^3-3abc+a^3+c^3)=0$$

$$a^3+b^3+c^3-3abc=0 \;\wedge\; 4b=0 \qquad 4\neq 0$$

$$a^3+b^3+c^3=3abc \;\wedge\; b=0$$

Thus roots will be Equal if

$b=0$ or $a^3+b^3+c^3=3abc$

Simultaneous Equations:

"The equations which have same solution for whole system are called Simultaneous Equations."

Exercise 4.8

Solve System of Equations:

(1) $2x-y=4$ —— (i)

$2x^2-4xy-y^2=6$ —— (ii)

From (i) $y=2x-4$

Putting in (ii), we have

$$2x^2-4x(2x-4)-(2x-4)^2=6$$

$$2x^2-8x^2+16x-(4x^2+16-16x)=6$$

$$2x^2-8x^2+16x-4x^2-16+16x-6=0$$

$$-10x^2+32x-22=0$$

$$-2\{5x^2-16x+11\}=0 \qquad -2\neq 0$$

$$5x^2-16x+11=0$$

$$5x^2-11x-5x+11=0$$

$$(5x-11)(x-1)=0$$

$x-1=0 \;\wedge\; 5x-11=0$

$x=1$

$y=2x-4$

$y=2(1)-4$

$y=2-4=-2$

$y=-2$

$x=11/5$

$y=2x-4$

$y=2(11/5)-4$

$y=\frac{22}{5}-4=\frac{22-20}{5}$

$y=\frac{2}{5}$

Thus Solution Set is

$$\{(1,-2),(11/5,\,2/5)\}$$