$(y+3)(y-3)(y^2+9)=0$

$y+3=0 \wedge y-3=0 \wedge y^2+9=0$

$y=-3 \wedge y=3 \wedge y^2=9i^2$

$y=\pm 3 \wedge y=\pm 3i$

Thus Solution Set is

$\{0, \pm 3, \pm 3i\}$

(iii) $x^3+x^2+x+1=0$

$x^2(x+1)+1(x+1)=0$

$(x+1)(x^2+1)=0$

$x+1=0 \wedge x^2+1=0$

$x=-1 \wedge x^2=i^2$

$x=-1 \wedge x=\pm i$

Thus Solution Set is

$\{-1, \pm i\}$

(iv) $5x^5-5x=0$

$5x[x^4-1]=0$

$5x=0 \wedge x^4-1=0$

$x=0 \quad 5\neq 0 \wedge (x^2-1)(x^2+1)=0$

$(x+1)(x-1)(x^2+1)=0$

$x+1=0 \wedge x-1=0 \wedge x^2+1=0$

$x=-1 \wedge x=1 \wedge x^2=i^2$

$x=\pm 1 \wedge x=\pm i$

Thus Solution Set is

$\{0, \pm 1, \pm i\}$

Polynomial:

A polynomial is an expression of variables and constants in the form of

$a_n x^n + a_{n-1} x^{n-1} + \dots + a_3 x^3 + a_2 x^2 + a_1 x + a_0$

where n is a whole number. Highest power of variable is called the degree of Polynomial.

Theorems of Polynomials:-

(i) Remainder Theorem:-

Statement:-

If $P(x)$ a polynomial of degree $n \geq 1$ is divided by $(x-a)$ till no term contain x exist in the remainder, then $P(a)$ is the remainder of $P(x)$.

Proof:- Let us divide $P(x)$ by $(x-a)$ and a unique quotient $Q(x)$ and a unique remainder R exists such as

$$P(x) = (x-a)Q(x) + R \quad \text{\_\_\_\_ (i)}$$

$\Rightarrow \quad x-a=0 \Rightarrow x=a$

Putting in (i)

$P(a) = (a-a)Q(x) + R$

$P(a) = R \qquad$ (Proved)

If $x-a$ is a factor of $P(x)$ then $R=0$

(ii) Factor Theorem:-

Statement:-

It states that $x-a$ will be a factor of $P(x)$ polynomial if and only if remainder of polynomial is zero

i.e. $P(a) = 0$

Proof:- Let $(x-a)$ be a divisor of polynomial $P(x)$ having quotient $Q(x)$ and Remainder R then by Remainder theorem

$P(x) = (x-a)\,Q(x) + R$

$\because$ $(x-a)$ will be factor of $P(x)$ if $R=0$

$P(x) = (x-a)\,Q(x) + 0$

$P(x) = (x-a)\,Q(x)$

Thus $(x-a)$ is a factor of $P(x)$.

Conversely, If $(x-a)$ is a factor of $P(x)$ then

$R = P(a) = 0$

This proves the theorem.

Synthetic Division:-

This is a short cut method of division of two polynomials called Synthetic Division.

Explaination in Questions.

Exercise: 4.5 (33)

Using remainder theorem, find the remainder.

Q.1 x^2+3x+7 , $x+1$

Let $P(x) = x^2+3x+7$

and let $x+1=0 \Rightarrow x=-1$

Using Remainder theorem

$P(-1) = R$

$R = (-1)^2+3(-1)+7$

$R = 1-3+7$

$R = 8-3 = 5$ Ans

Q.2 x^3-x^2+5x+4 , $x-2$

Let $P(x) = x^3-x^2+5x+4$

and let $x-2=0 \Rightarrow x=2$

Using Remainder theorem

$P(2) = R$

$R = (2)^3-(2)^2+5(2)+4$

$R = 8-4+10+4$

$R = 18$ Ans.

Q.3 $3x^4+4x^3+x-5$, $x+1$

Let $P(x) = 3x^4+4x^3+x-5$

let $x+1=0 \Rightarrow x=-1$

Using Remainder theorem

$P(-1) = R \Rightarrow R = 3(-1)^4+4(-1)^3+(-1)-5$

$R = 3-4-1-5 \Rightarrow R = -7$

$R = -7$ Ans

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Q.4 x^3-2x^2+3x+3 , $x-3$

Let $P(x)=x^3-2x^2+3x+3$

let $x-3=0 \Rightarrow x=3$

Using Remainder Theorem

$R = P(3)$

$R = (3)^3-2(3)^2+3(3)+3$

$R = 27-18+9+3$

$R = 21$ Ans.

Q.5 x^2+4x-5 , $x-1$

Let $P(x)=x^2+4x-5$ let $x-1=0$

$x=1$

Using factor theorem

it will be factor if $R=P(1)=0$

$R=(1)^2+4(1)-5$

$R=1+4-5=5-5=0$

Thus $x-1$ is a factor of $P(x)$.

Q.6 x^3+x^2-7x+1 , $x-2$

$P(x)=x^3+x^2-7x+1$ let $x-2=0$

$x=2$

Using factor theorem

$R=(2)^3+(2)^2-7(2)+1$

$R=8+4-14+1=13-14=-1$

$R\neq 0$

Thus $x-2$ is not a factor of $P(x)$.

Q.7 $2w^3+w^2-4w+7$, $w+2$

$P(w)=2w^3+w^2-4w+7$

$w+2=0 \Rightarrow w=-2$

Using factor theorem

$R=2(-2)^3+(-2)^2-4(-2)+7$

$R=-16+4+8+7$

$R=19-16=3\neq 0$

Thus $w+2$ is not a factor of $P(w)$.

Q.8 x^n-a^n , $x-a$ $n\in Z^+$

$P(x)=x^n-a^n$ let $x-a=0$

$x=a$

Using factor theorem

$R=(a)^n-a^n$

$R=0$

Thus $x-a$ is a factor of $P(x)$.

Q.9 x^n+a^n , $x+a$

Let $P(x)=x^n+a^n$

let $a+x=0 \Rightarrow x=-a$

Using factor theorem

$R=P(-a)$

$R=(-a)^n+a^n$ $\because n$ is odd

$R=0$

Thus $x+a$ is a factor of $P(x)$

Q.10 $x^4+2x^3+kx^2+3$, let $x-2=0$

$x=2$

$R=1$ $k=?$

Using Remainder theorem

$R=P(2)$

$1=(2)^4+2(2)^3+k(2)^2+3$

$1=16+16+4k+3$

$4k=-34 \Rightarrow k=\frac{-17}{2}$ Ans.

Q.11 $P(x) = x^3 + 2x^2 + Kx + 4$

Let $x - 2 = 0 \Rightarrow x = 2$

$R = 14 \qquad K = ?$

Using Remainder theorem

$R = P(2)$

$14 = (2)^3 + (2)^2 2 + K(2) + 4$

$14 = 8 + 8 + 2K + 4$

$2K = -6$

$K = -3$ Ans.

Q.12 $P(x) = x^3 - 7x + 6 = 0 \qquad x = 2$

$x - 2 = 0$

	1	0	-7	6
2		2	4	-6
	1	2	-3	0 — Remainder

Thus $x = 2$ is solution of $P(x)$

$P(x) = (x-2)(x^2 + 2x - 3)$

$P(x) = (x-2)(x^2 + 3x - x - 3)$

$P(x) = (x-2)\{x(x+3) - 1(x+3)\}$

$P(x) = (x-2)\{(x+3)(x-1)\}$

$P(x) = (x-1)(x-2)(x+3)$

Thus Polynomial is factorized.

Q.13 $P(x) = x^3 - 28x - 48 = 0 \qquad x = -4$

$x + 4 = 0$

	1	0	-28	-48
-4		-4	16	+48
	1	-4	-12	0 Remainder

Thus $x = -4$ is root of $P(x)$

$P(x) = (x+4)(x^2 - 4x - 12)$

$P(x) = (x+4)\{x^2 - 6x + 2x - 12\}$

$P(x) = (x+4)\{x(x-6) + 2(x-6)\}$

$P(x) = (x+4)(x-6)(x+2)$

Thus polynomial is factorized.

Q.14 $P(x) = 2x^4 + 7x^3 - 4x^2 - 27x - 18$

$x = 2, -3$

$(x-2), (x+3)$

	2	7	-4	-27	-18
2		4	22	36	18
	2	11	18	9	0 — Remainder
-3		-6	-15	-9	
	2	5	3	0 — Remainder	

Thus $x = 2, -3$ are the roots of $P(x)$

$P(x) = (x-2)(x+3)(2x^2 + 5x + 3)$

$P(x) = (x-2)(x+3)\{2x^2 + 3x + 2x + 3\}$

$P(x) = (x-2)(x+3)\{x(2x+3) + 1(2x+3)\}$

$P(x) = (x-2)(x+3)(2x+3)(x+1)$

Thus polynomial is Completely factorized

Q.15 $P(x) = x^3 + px^2 + qx + 6$

$(x+1), (x-2)$ are factors of $P(x)$

$x = -1, 2$

$p = ? \quad q = ?$

	1	p	q	6
-1		-1	1-p	p-q-1
	1	p-1	1-p+q	p-q+5 — Remainder
2		2	2p+2	
	1	p+1	p+q+3 — Remainder	

$\because (x+1)(x-2)$ are factors of $P(x)$

so remainder must be zero.

$p - q + 5 = 0 \quad \text{—} (1)$

$p + q + 3 = 0 \quad \text{—} (2)$

Adding ① and ②

$$2P+8=0 \Rightarrow \boxed{P=-4}$$

Subtracting ① and ②

$$-2q+2=0 \Rightarrow \boxed{q=1}$$

Q.16 $P(x)=x^3-4x^2+ax+b$

$x=-2,2 \qquad a=?\ b=?$

	1	-4	a	b
-2		-2	12	-2a-24
	1	-6	a+12	-2a+b-24 ← Remainder
2		2	-8	
	1	-4	a+4 ←	

$\because$ 2, -2 are the roots of $P(x)$ so remainder must be zero.

$$a+4=0 \Rightarrow \boxed{a=-4}$$

$$-2a+b-24=0$$

$$-2(-4)+b=24$$

$$8+b=24 \Rightarrow \boxed{b=16}$$

Relation between Roots and Coefficients of a Quadratic Equation:-

Let α, β be the roots of $ax^2+bx+c=0$

$$\alpha=\frac{-b+\sqrt{b^2-4ac}}{2a} \qquad \beta=\frac{-b-\sqrt{b^2-4ac}}{2a}$$

Sum of roots $= \alpha+\beta$

$$\alpha+\beta=\frac{-b+\sqrt{b^2-4ac}}{2a}+\frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$\alpha+\beta=\frac{-b+\sqrt{b^2-4ac}-b-\sqrt{b^2-4ac}}{2a}$$

$$\alpha+\beta=\frac{-2b}{2a} \Rightarrow \alpha+\beta=-\frac{b}{a}$$

$$S=\alpha+\beta=\frac{-b}{a}=-\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2}$$

Product of roots $= \alpha\beta$

$$\alpha\beta=\left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\cdot\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right)$$

$$\alpha\cdot\beta=\frac{(-b)^2-\left(\sqrt{b^2-4ac}\right)^2}{4a^2}$$

$$\alpha\beta=\frac{\cancel{b^2}-\cancel{b^2}+4ac}{4a^2}=\frac{4ac}{4a^2}$$

$$\alpha\beta=\frac{c}{a}=\frac{\text{Constant term}}{\text{Coefficient of } x^2}$$

$$\alpha\beta=P=\frac{c}{a}$$

Formation of an Equation

Let $(x-\alpha)(x-\beta)=0$ has roots α, β

$$\Rightarrow x^2-(\alpha+\beta)x+\alpha\beta=0$$

$$x^2-Sx+P=0$$

where, S = sum of roots

and P = Product of roots.

Exercise 4.6

Q.1 If α, β are the roots of $3x^2-2x+4=$

find the values of:

$$S=\alpha+\beta=-\frac{b}{a}=\frac{+2}{3}=\frac{2}{3}$$

$$P=\alpha\beta=\frac{c}{a}=\frac{4}{3}$$

(i) $\frac{1}{\alpha^2}+\frac{1}{\beta^2}=?$

$$\frac{1}{\alpha^2}+\frac{1}{\beta^2}=\frac{\beta^2+\alpha^2}{\alpha^2\beta^2}$$

$$\frac{1}{\alpha^2}+\frac{1}{\beta^2}=\frac{\alpha^2+\beta^2+2\alpha\beta-2\alpha\beta}{(\alpha\beta)^2}$$

$$\frac{1}{\alpha^2}+\frac{1}{\beta^2}=\frac{(\alpha+\beta)^2-2(\alpha\beta)}{(\alpha\beta)^2}$$