# Cube Roots of Unity:-



Let $x$ is the cube root of Unity

$$x = (1)^{1/3} \Rightarrow x^3 = 1$$

$$x^3 - 1 = 0$$

$$(x-1)(x^2+x+1) = 0$$

$$x-1=0 \quad \wedge \quad x^2+x+1=0$$

$$x = 1 \quad \text{——} \ (1)$$

$$x^2+x+1=0$$

$$x = \frac{-1 \pm \sqrt{1-4(1)(1)}}{2}$$

$$x = \frac{-1 \pm \sqrt{-3}}{2} \Rightarrow x = \frac{-1 \pm \sqrt{3}\,i}{2}$$

Thus Cube roots of Unity are

$$1,\ \frac{-1+\sqrt{3}\,i}{2},\ \frac{-1-\sqrt{3}\,i}{2}$$

* The roots containing $i$ are called Complex or imaginary roots of Unity.

## Properties of Cube roots of Unity.

(1) Each Complex root is Square of other Complex root.

$$\left(\frac{-1+\sqrt{3}\,i}{2}\right)^2 = \frac{(-1)^2+(\sqrt{3}\,i)^2+2(-1)(\sqrt{3}\,i)}{4}$$

$$\left(\frac{-1+\sqrt{3}\,i}{2}\right)^2 = \frac{1-3-2\sqrt{3}\,i}{4}$$

$$\left(\frac{-1+\sqrt{3}\,i}{2}\right)^2 = \frac{-2-2\sqrt{3}\,i}{4}$$

$$\left(\frac{-1+\sqrt{3}\,i}{2}\right)^2 = \frac{\cancel{2}}{\cancel{2}}\left[\frac{-1-\sqrt{3}\,i}{2}\right]$$

$$\left(\frac{-1+\sqrt{3}\,i}{2}\right)^2 = \left(\frac{-1-\sqrt{3}\,i}{2}\right) \quad \text{(Proved)}$$

Let $\omega = \frac{-1+\sqrt{3}\,i}{2}$ ($\omega$ = omega)

then $\omega^2 = \frac{-1-\sqrt{3}\,i}{2}$

(2) Sum of Cube roots of Unity is Zero.

$$1+\omega+\omega^2 = 1+\left(\frac{-1+\sqrt{3}\,i}{2}\right)+\left(\frac{-1-\sqrt{3}\,i}{2}\right)$$

$$1+\omega+\omega^2 = \frac{2-1+\sqrt{3}\,i-1-\sqrt{3}\,i}{2}$$

$$1+\omega+\omega^2 = \frac{0}{2} = 0$$

$$1+\omega+\omega^2 = 0 \quad \text{(Proved)}$$

(3) Product of Cube roots of Unity is 1

$$1\cdot\omega\cdot\omega^2 = 1\cdot\left(\frac{-1+\sqrt{3}\,i}{2}\right)\cdot\left(\frac{-1-\sqrt{3}\,i}{2}\right)$$

$$\omega^3 = \frac{(-1)^2-(\sqrt{3}\,i)^2}{4} \Rightarrow \omega^3 = \frac{(-1)^2-(3i^2)}{4}$$

$$\omega^2 = \frac{1-(-3)}{4} \Rightarrow \omega^3 = \frac{1+3}{4}$$

$$\omega^3 = \frac{4}{4} \Rightarrow \omega^3 = 1 \quad \text{(Proved).}$$

## Fourth Roots of Unity

let $x = 1^{1/4} \Rightarrow x^4 = 1$

$$x^4-1=0 \Rightarrow (x^2-1)(x^2+1)=0$$

$$(x+1)(x-1)(x^2+1)=0$$

$$x+1=0 \ \wedge\ x-1=0 \ \wedge\ x^2+1=0$$

$$x=-1 \ \wedge\ x=1 \ \wedge\ x^2=i^2 \Rightarrow x=\pm$$

Thus fourth roots of Unity a

$$1,\ -1,\ i,\ -i$$

Properties of fourth roots of Unity.

(1) Sum of all the fourth roots of unity is zero.

(2) The product of all the fourth roots of Unity is $-1$

(3) The real fourth roots of unity are additive inverse of eachother.

(4) The Complex/Imaginary 4th roots of unity are Conjugate to each other.

# Exercise: 4.4

**Q.1 Find the Cube roots of**

(i) Let $x = (8)^{1/3} \Rightarrow x^3 = 8$

$x^3 - 8 = 0 \Rightarrow x^3 - (2)^3$

$\Rightarrow (x-2)(x^2+2x+4) = 0$

$x-2=0 \quad \wedge \quad x^2+2x+4=0$

$x=2 \quad \wedge \quad x = \frac{-2\pm\sqrt{4-16}}{2}$

$x=2 \quad \wedge \quad x = \frac{-2\pm\sqrt{-12}}{2}$

$x=2 \quad \wedge \quad x = \frac{-2\pm 2\sqrt{3}i}{2} = 2\left(\frac{-1\pm\sqrt{3}i}{2}\right)$

$x=2 \quad \wedge \quad x = 2\left(\frac{-1+\sqrt{3}i}{2}\right), 2\left(\frac{-1-\sqrt{3}i}{2}\right)$

Thus Cube roots of 8 are

$2, \; 2\omega, \; 2\omega^2$

(ii) let $x = (-8)^{1/3} \Rightarrow x^3 = -8$

$x^3+8=0 \Rightarrow x^3+(2)^3=0$ **(28)**

$(x+2)(x^2-2x+4)=0$

$x+2=0 \quad \wedge \quad x^2-2x+4=0$

$x=-2 \quad \wedge \quad x = \frac{2\pm\sqrt{4-16}}{2}$

$x=-2 \quad \wedge \quad x = -2\left(\frac{-1+\sqrt{3}i}{2}\right), -2\left(\frac{-1-\sqrt{3}i}{2}\right)$

$x=-2 \quad \wedge \quad x = -2\omega, -2\omega^2$

Thus Cube roots of $-8$ are

$-2, \; -2\omega, \; -2\omega^2$.

(iii) let $x = (27)^{1/3} \Rightarrow x^3 = 27$

$x^3-27=0 \Rightarrow x^3-3^3=0$

$(x-3)(x^2+3x+9)=0$

$x-3=0 \quad \wedge \quad x^2+3x+9=0$

$x=3 \quad \wedge \quad x = \frac{-3\pm\sqrt{9-36}}{2}$

$x=3 \quad \wedge \quad x = 3\left(\frac{-1+\sqrt{3}i}{2}\right), 3\left(\frac{-1-\sqrt{3}i}{2}\right)$

Thus Cube roots of 27 are

$3, \; 3\omega, \; 3\omega^2$ Similarly (iv)

(v) let $x = (64)^{1/3} \Rightarrow x^3 = 64$

$x^3-64=0 \Rightarrow x^3-4^3=0$

$(x-4)(x^2+4x+16)=0$

$x-4=0 \quad \wedge \quad x^2+4x+16=0$

$x=4 \quad \wedge \quad x = \frac{-4\pm\sqrt{16-64}}{2}$

$x=4 \quad \wedge \quad x = 4\left(\frac{-1+\sqrt{3}i}{2}\right) = 4\left(\frac{-1-\sqrt{3}i}{2}\right)$

Thus Cube roots of 64 are

$4, \; 4\omega, \; 4\omega^2$

## Q.2 Evaluate

(i) $(1+\omega-\omega^2)^8$

$= (1+\omega-\omega^2)^8$

$= (-\omega^2-\omega^2)^8 \quad \because 1+\omega+\omega^2=0, \quad 1+\omega=-\omega^2$

$= (-2\omega^2)^8$

$= (-2)^8 \cdot (\omega^2)^8$

$= 256 \cdot \omega^{16}$

$= 256\,(\omega^3)^5 \cdot \omega$

$= 256\,(1)^5 \omega \quad \because \omega^3=1$

$= 256\,\omega$ **Ans**

(ii) $\omega^{28}+\omega^{29}+1$

$= (\omega^3)^9 \cdot \omega + (\omega^3)^9 \cdot \omega^2 + 1$

$= (1)^9 \cdot \omega + (1)^9 \cdot \omega^2 + 1 \quad \because \omega^3=1$

$= \omega + \omega^2 + 1$

$= 1+\omega+\omega^2 \quad \because 1+\omega+\omega^2=0$

$= 0$ **Ans**

(iii) $(1+\omega-\omega^2)(1-\omega+\omega^2)$

$= (-\omega^2-\omega^2)(-\omega+1+\omega)$

$= (-2\omega^2)(-\omega-\omega) \quad \because 1+\omega+\omega^2=0, \quad 1+\omega=-\omega^2, \quad 1+\omega^2=-\omega$

$= (-2\omega^2)(-2\omega)$

$= 4\omega^3 \quad \because \omega^3=1$

$= 4(1) = 4$ **Ans.**

(iv) $\left(\frac{-1+\sqrt{-3}}{2}\right)^9 + \left(\frac{-1-\sqrt{-3}}{2}\right)^7$

we know that

$\omega = \frac{-1+\sqrt{3}\,i}{2} \qquad \omega^2 = \frac{-1-\sqrt{3}\,i}{2}$

where $i=\sqrt{-1}$

$= \omega^9 + (\omega^2)^7$

$= \omega^9 + \omega^{14}$

$= (\omega^3)^3 + (\omega^3)^4 \cdot \omega^2 \quad \because \omega^3=1$

$= (1)^3 + (1)^4 \cdot \omega^2$

$= 1 + \omega^2 \quad \because 1+\omega+\omega^2=0, \quad 1+\omega^2=-\omega$

$= -\omega$ **Ans.**

(v) $(-1+\sqrt{-3})^5 + (-1-\sqrt{-3})^5$

$= \left[2\left(\frac{-1+\sqrt{-3}}{2}\right)\right]^5 + \left[2\left(\frac{-1-\sqrt{-3}}{2}\right)\right]^5$

$= (2\omega)^5 + (2\omega^2)^5$

$= (2)^5\omega^5 + (2)^5 \cdot \omega^{10}$

$= 2^5\left[\omega^5+\omega^{10}\right]$

$= 32\left[\omega^3 \cdot \omega^2 + (\omega^3)^3 \cdot \omega\right]$

$= 32\left[1 \cdot \omega^2 + (1)^3 \cdot \omega\right] \quad \because \omega^3=1$

$= 32\left[\omega^2+\omega\right] \quad \because 1+\omega+\omega^2=0, \quad \omega+\omega^2=-1$

$= 32[-1]$

$= -32$ **Ans**

**Example(2) Prove that**

$(-1+\sqrt{-3})^4 + (-1-\sqrt{-3})^4 = -16$

$L.H.S = (-1+\sqrt{-3})^4 + (-1-\sqrt{-3})^4$

$= \left[2\left(\frac{-1+\sqrt{-3}}{2}\right)\right]^4 + \left[2\left(\frac{-1-\sqrt{-3}}{2}\right)\right]^4$

$= (2\omega)^4 + (2\omega^2)^4$

$= 16\omega^4 + 16\omega^8 = 16\left[\omega^3 \cdot \omega + (\omega^3)^2 \cdot \omega^2\right]$

$= 16\left[\omega+\omega^2\right] \quad \because \omega^3=1, \quad \omega+\omega^2=-1$

$= 16[-1]$

$= -16$ RHS

**Q.3 Show that:**

**(i)** $x^3-y^3=(x-y)(x-\omega y)(x-\omega^2 y)$

$R.H.S.=(x-y)(x-\omega y)(x-\omega^2 y)$

$=(x-y)\{x^2-\omega^2 xy-\omega xy+\omega^3 y^2\}$

$=(x-y)\{x^2-xy(\omega+\omega^2)+\omega^3 y^2\}$

$\because \omega^3=1$ and $1+\omega+\omega^2=0 \Rightarrow \omega+\omega^2=-1$

$=(x-y)\{x^2-xy(-1)+(1)y^2\}$

$=(x-y)(x^2+xy+y^2)$

$=x^3-y^3=L.H.S.$

**(ii)** $x^3+y^3+z^3-3xyz=(x+y+z)(x+\omega y+\omega^2 z)\cdot(x+\omega^2 y+\omega z)$

$R.H.S=(x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z)$

$=(x+y+z)\{x^2+\omega^2 xy+\omega xz+x\omega y+\omega^3 y^2$
$+\omega^2 yz+\omega^2 xz+\omega^4 yz+\omega^3 z^2\}$

$=(x+y+z)\{x^2+\omega^3 y^2+\omega^3 z^2+xy(\omega+\omega^2)$
$+yz(\omega+\omega^2)+zx(\omega+\omega^2)\}$

$\because \omega^3=1$, $1+\omega+\omega^2=0 \Rightarrow \omega+\omega^2=-1$

$=(x+y+z)\{x^2+y^2+z^2+xy(-1)+yz(-1)+zx(-1)\}$

$=(x+y+z)(x^2+y^2+z^2-xy-yz-zx)$

$=x^3+y^3+z^3-3xyz=L.H.S$

**(iii)** $(1+\omega)(1+\omega^2)(1+\omega^4)\cdots$ 2n factors $=1$

$L.H.S=(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)\cdots$ 2n terms

$=(1+\omega)(1+\omega^2)(1+\omega^3\cdot\omega)+(1+\omega^6\cdot\omega^2)\cdots$ 2n terms

$\because \omega^3=1$

$=(1+\omega)(1+\omega^2)(1+\omega)(1+\omega^2)\cdots$ 2n terms

$=[(1+\omega)(1+\omega^2)][(1+\omega)(1+\omega^2)]\cdots$ n terms.

Here two terms are collected together.

$\because a\cdot a\cdot a\cdots$ n terms $=a^n$

$=[(1+\omega)(1+\omega^2)]^n$

$=(1+\omega^3+\omega+\omega^2)^n$

$=(1+\omega+\omega^2+\omega^3)^n$

$\because 1+\omega+\omega^2=0$ and $\omega^3=1$

$=[0+1]^n$

$=(1)^n=1$ RHS.

**Example(1) Prove that:**

$(x^3+y^3)=(x+y)(x+\omega y)(x+\omega^2 y)$

$RHS=(x+y)(x+\omega y)(x+\omega^2 y)$

$=(x+y)\{x^2+xy\omega^2+xy\omega+\omega^3 y^2\}$

$=(x+y)\{x^2+xy(\omega+\omega^2)+\omega^3 y^2\}$

$\because \omega^3=1$ $1+\omega+\omega^2=0 \Rightarrow \omega+\omega^2=-1$

$=(x+y)\{x^2+xy(-1)+(1)y^2\}$

$=(x+y)\{x^2-xy+y^2\}$

$=x^3+y^3=L.H.S.$

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**Q.4** This Question is exactly the proof of Cube roots of Unit other than $(x-1)$. Do yourself.

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**Q.5** Let $x=(-1)^{1/3}$

$x^3=-1 \Rightarrow x^3+1=0$

$(x+1)(x^2-x+1)=0$

$x+1=0 \wedge x^2-x+1=0$

$x=-1$ —— (i)

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$x^2 - x + 1 = 0$

$$x = \frac{1 \pm \sqrt{1-4}}{2} \Rightarrow x = \frac{1 \pm \sqrt{-3}}{2}$$

$$x = \frac{1+\sqrt{-3}}{2}, \frac{1-\sqrt{-3}}{2}$$

$$x = -\omega, -\omega^2$$

Thus Cube roots of $-1$ are

$$-1, -\omega, -\omega^2$$

or $-1, \frac{1+\sqrt{-3}}{2}, \frac{1-\sqrt{-3}}{2}$

Thus Complex Cube roots are

$$\frac{1+\sqrt{3}i}{2}, \frac{1-\sqrt{3}i}{2}$$

Now Proof of

$$\left(\frac{1+\sqrt{-3}}{2}\right)^9 + \left(\frac{1-\sqrt{-3}}{2}\right)^9 = -2$$

<u>LHS</u> $= \left(\frac{1+\sqrt{-3}}{2}\right)^9 + \left(\frac{1-\sqrt{-3}}{2}\right)^9$

$$= (-\omega)^9 + (-\omega^2)^9$$

$$= -\omega^9 + (-\omega^{18})$$

$$= -\omega^9 - \omega^{18}$$

$$= -1\left((\omega^3)^3 + (\omega^3)^6\right) \quad \because \omega^3 = 1$$

$$= -1(1^3 + 1^6)$$

$$= -1(1+1) = -1(2)$$

$$= -2 = RHS$$

**Q.6** This Question will be solved after **4.6** so see after 4.6 Ex.

**Q.7 Find The four fourth roots of:**

**(i)** Let $x^4 = 16 \Rightarrow x^4 - 16 = 0$

$$x^4 - 2^4 = 0$$

$$(x^2-4)(x^2+4) = 0$$

$$(x-2)(x+2)(x^2+4) = 0$$

$$x-2=0 \wedge x+2=0 \wedge x^2+4=0$$

$$x=2 \wedge x=-2 \wedge x^2 = 4i^2$$

$$x=2 \wedge x=-2 \wedge x = \pm 2i$$

Thus four fourth roots of 16 are

$$\{\pm 2, \pm 2i\}$$

Similarly solve other two parts.

**Q.8 Solve the following Equations.**

**(i)** $2x^4 - 32 = 0$

$$\Rightarrow 2(x^4 - 16) = 0$$

$$x^4 - 2^4 = 0 \qquad 2 \neq 0$$

$$(x^2-4)(x^2+4) = 0$$

$$(x-2)(x+2)(x^2+4) = 0$$

$$x-2=0 \wedge x+2=0 \wedge x^2+4=0$$

$$x=2 \wedge x=-2 \wedge x^2 = 4i^2$$

$$x = \pm 2 \wedge x = \pm 2i$$

Thus Solution is

$$\{\pm 2, \pm 2i\}$$

**(iii)** $3y^5 - 243y = 0$

$$\Rightarrow 3y(y^4 - 81) = 0$$

$$3y = 0 \wedge y^4 - 81 = 0$$

$$y = 0 \quad 3 \neq 0 \wedge (y^2-9)(y^2+9) = 0$$

$(y+3)(y-3)(y^2+9)=0$

$y+3=0 \wedge y-3=0 \wedge y^2+9=0$

$y=-3 \wedge y=3 \wedge y^2=9i^2$

$y=\pm 3 \wedge y=\pm 3i$

Thus Solution Set is

$\{0, \pm 3, \pm 3i\}$

**(iii)** $x^3+x^2+x+1=0$

$x^2(x+1)+1(x+1)=0$

$(x+1)(x^2+1)=0$

$x+1=0 \wedge x^2+1=0$

$x=-1 \wedge x^2=i^2$

$x=-1 \wedge x=\pm i$

Thus Solution Set is

$\{-1, \pm i\}$

**(iv)** $5x^5-5x=0$

$5x[x^4-1]=0$

$5x=0 \wedge x^4-1=0$

$x=0 \quad 5\neq 0 \wedge (x^2-1)(x^2+1)=0$

$(x+1)(x-1)(x^2+1)=0$

$x+1=0 \wedge x-1=0 \wedge x^2+1=0$

$x=-1 \wedge x=1 \wedge x^2=i^2$

$x=\pm 1 \wedge x=\pm i$

Thus Solution Set is

$\{0, \pm 1, \pm i\}$ TAHIR





## Polynomial:



A polynomial is an expression of variables and constants in the form of

$a_n x^n + a_{n-1}x^{n-1} + \dots + a_3x^3 + a_2x^2 + a_1x + a_0$

where $n$ is a whole number. Highest power of variable is called the degree of Polynomial.

## Theorems of Polynomials:-

### (i) Remainder Theorem:-

**Statement:-**

If $P(x)$ a polynomial of degree $n \geq 1$ is divided by $(x-a)$ till no term contain $x$ exist in the remainder, then $P(a)$ is the remainder of $P(x)$.

**Proof:-** Let us divide $P(x)$ by $(x-a)$ and a unique quotient $Q(x)$ and a unique remainder $R$ exists such as

$$P(x) = (x-a)\,Q(x) + R \quad \text{---(i)}$$

$\Rightarrow x-a=0 \Rightarrow x=a$

Putting in (i)

$P(a) = (a-a)\,Q(x) + R$

$P(a) = R$ (Proved)

If $x-a$ is a factor of $P(x)$ then $R=0$