Quadratic Equations.

Quadratic Equation :-

"The equation in which highest degree of variable is 2 is called 2nd degree or quadratic equation."

* $ax^2 + bx + c = 0$ is the standard form of quadratic equation in one variable x where a, b, c are arbitrary constants and $a \neq 0$ * If $b = 0$ then equation is called pure quadratic.
* Quadratic Equation may be algebric or trigonometric.

Solution of Quadratic Equations :-

* The particular values which satisfy the equation are called "Roots of Equation."
* The process of finding the roots of a quadratic equation is called "Solution of Quadratic Equation."
* The set containing all the roots of a quadratic equation is called "Solution Set."
* There are mainly three methods used to solve a quadratic equation in one variable x.

  (i) By Factorization method.

  (ii) By Completing the Square method.

  (iii) By Quadratic Formula.

* The quadratic Equation may be in one or two variables or more than two variables.

Tahir Mahmood
M.Sc. (Math)
Mob No: 0345-6510779

* The equation containing identical (same) degree of every term is called Homogeneous Equation.

  i.e $ax^2 + bxy + cy^2 = 0$

Derivation of Quadratic Formula:

The standard quadratic Equation is $ax^2+bx+c=0$

The Quadratic Formula can be derived using Completing the square method

$$ax^2+bx+c=0$$

Step (1) $\Longrightarrow$ $ax^2+bx=-c$ (Transfering constant towards the other side)

Step (2) $\Longrightarrow$ Dividing both the sides by the coefficient of x^2, we have

$$x^2+\frac{b}{a}x=\frac{-c}{a}$$

Step (3) $\Longrightarrow$ Adding the square of half of the coefficient of x, i.e $\left(\frac{b}{2a}\right)^2$

$$x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=\frac{-c}{a}+\left(\frac{b}{2a}\right)^2$$

$$x^2+2(x)\left(\frac{b}{2a}\right)+\left(\frac{b}{2a}\right)^2=\frac{-c}{a}+\frac{b^2}{4a^2}$$

$$\left(x+\frac{b}{2a}\right)^2=\frac{-4ac+b^2}{4a^2}$$

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$

Taking Square root on both sides, we have

$$\left(x+\frac{b}{2a}\right)=\frac{\pm\sqrt{b^2-4ac}}{2a}$$

$$\Longrightarrow\quad x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$\boxed{x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}}$$ (Proved).

This is called quadratic Formula which is the solution of $ax^2+bx+c=0$

* To solve the equations by Quadratic Formula, we compare the equation with $ax^2+bx+c=0$

Solve the followings by factorization:

① $3x^2+4x+1=0$

$= 3x^2+3x+x+1=0$

$3x(x+1)+1(x+1)=0$

$(x+1)(3x+1)=0$

$x+1=0 \quad \wedge \quad 3x+1=0$

$x=-1 \quad \wedge \quad x=\frac{-1}{3}$

$S.S.=\{-1, \frac{-1}{3}\}$

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⑤ $x(x+7)=(2x-1)(x+4)$

$x^2+7x=2x^2+8x-x-4$

$\Rightarrow 2x^2-x^2+8x-x-7x-4=0$

$x^2-4=0$

$(x+2)(x-2)=0$

$x+2=0 \quad \wedge \quad x-2=0$

$x=-2 \quad \wedge \quad x=2$

$S.S=\{-2, 2\}$

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⑥ $\frac{x}{x+1}+\frac{x+1}{x}=\frac{5}{2}\ ; \quad x \neq 0, -1$

$\frac{x}{x+1}+\frac{x+1}{x}=\frac{5}{2}$

$\frac{x^2+(x+1)^2}{x(x+1)}=\frac{5}{2}$

$\frac{x^2+x^2+2x+1}{x^2+x}=\frac{5}{2}$

$2\{x^2+x^2+2x+1\}=5\{x^2+x\}$

$2\{2x^2+2x+1\}=5\{x^2+x\}$

$4x^2+4x+2=5x^2+5x$

$\Rightarrow 5x^2-4x^2+5x-4x-2=0$

$\Rightarrow x^2+x-2=0$

$\Rightarrow x^2+2x-x-2=0$

$\Rightarrow x(x+2)-1(x+2)=0$

$\Rightarrow (x-1)(x+2)=0$

$\Rightarrow x-1=0 \quad \wedge \quad x+2=0$

$\Rightarrow x=1 \quad \wedge \quad x=-2$

$S.S=\{-2, 1\}$

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⑦ $\frac{1}{x+1}+\frac{2}{x+2}=\frac{7}{x+5}$

$\frac{(x+2)+2(x+1)}{(x+1)(x+2)}=\frac{7}{x+5}$

$\frac{x+2+2x+2}{x^2+2x+x+2}=\frac{7}{x+5}$

$\frac{(3x+4)}{(x^2+3x+2)}=\frac{7}{(x+5)}$

$(3x+4)(x+5)=7(x^2+3x+2)$

$3x^2+15x+4x+20=7x^2+21x+14$

$7x^2-3x^2+21x-19x+14-20=0$

$4x^2+2x-6=0$

$4x^2+6x-4x-6=0$

$2x(2x+3)-2(2x+3)=0$

$(2x-2)(2x+3)=0$

$2x-2=0 \quad \wedge \quad 2x+3=0$

$x=\frac{2}{2} \quad \wedge \quad x=\frac{-3}{2}$

$x=1 \quad \wedge \quad x=\frac{-3}{2}$

$S.S=\{1, \frac{-3}{2}\}$

(8) $\dfrac{a}{ax-1} + \dfrac{b}{bx-1} = a+b$

$$\frac{a}{(ax-1)} - b + \frac{b}{bx-1} - a = 0$$

$$\frac{a-abx+b}{ax-1} + \frac{b-abx+a}{bx-1} = 0$$

$$a+b-abx\left\{\frac{1}{ax-1} + \frac{1}{bx-1}\right\} = 0$$

$$(a+b-abx)\left\{\frac{bx-1+ax-1}{(ax-1)(bx-1)}\right\} = 0$$

Multiplying by $(ax-1)(bx-1)$

$$(a+b-abx)(ax+bx-2) = 0$$

$\Rightarrow a+b-abx = 0$ or $(a+b)x-2=0$

$\Rightarrow a+b = abx$ or $(a+b)x = 2$

$x = \dfrac{a+b}{ab}$ or $x = \dfrac{2}{a+b}$

$S.S = \left\{\dfrac{2}{a+b}, \dfrac{a+b}{ab}\right\}$

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(4) $x^2 - x = 2 \Rightarrow x^2 - x - 2 = 0$

$$x^2 - 2x + x - 2 = 0$$

$$x(x-2) + 1(x-2) = 0$$

$$(x-2)(x+1) = 0$$

$x-2 = 0$ or $x+1 = 0$

$x = 2 \qquad x = -1$

$S.S = \{-1, 2\}$

Solve the following be Completing the Square method:-

(9) $x^2 - 2x - 899 = 0$

$$x^2 - 2x = 899$$

$$x^2 - 2x + (1)^2 = 899 + (1)^2$$

Adding $(1)^2$ in both sides

$$x^2 - 2x + (1)^2 = 899 + 1$$

$$(x)^2 - 2(x)(1) + (1)^2 = 900$$

$$(x-1)^2 = (30)^2$$

Taking Square root, we have

$$(x-1) = \pm 30$$

$$x = 1 \pm 30$$

$x = 1+30 \quad \wedge \quad x = 1-30$

$x = 31 \quad \wedge \quad x = -29$

$S.S = \{31, -29\}$

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(14) $2x^2 + 12x - 110 = 0$

$$2x^2 + 12x = 110$$

Dividing both sides by 2

$$x^2 + 6x = 55$$

Adding $(3)^2$ in both sides

$$x^2 + 6x + (3)^2 = 55 + (3)^2$$

$$(x)^2 + 2(x)(3) + (3)^2 = 55 + 9$$

$$(x+3)^2 = 64$$

Taking Square root, we have

$$x+3 = \pm\sqrt{64}$$

$$x+3 = \pm 8$$

$$x = -3 \pm 8$$

$x = -3+8 \quad \wedge \quad x = -3-8$

$x = 5 \quad \wedge \quad x = -11$

$S.S = \{5, -11\}$

Students do remaining Parts themselves.

Solve the followings using Quadratic formula:-

(15) $5x^2 - 13x + 6 = 0$

Comping with $ax^2 + bx + c = 0$

$a = 5 \quad b = -13 \quad c = 6$

Using Quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-13) \pm \sqrt{(-13)^2 - 4(5)(6)}}{2(5)}$$

$$x = \frac{13 \pm \sqrt{169 - 120}}{10}$$

$$x = \frac{13 \pm \sqrt{49}}{10} \Rightarrow x = \frac{13 \pm 7}{10}$$

$$x = \frac{13+7}{10} \wedge x = \frac{13-7}{10}$$

$$x = \frac{20}{10} \wedge x = \frac{6}{10}$$

$$x = 2 \wedge x = 3/5$$

$$S.S = \{2, 3/5\}.$$

(17) $15x^2 + 2ax - a^2 = 0$

Comparing with $ax^2 + bx + c = 0$

$a = 15 \quad b = 2a \quad c = -a^2$

Using Quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-2a \pm \sqrt{4a^2 - 4(15)(-a^2)}}{2(15)}$$

$$x = \frac{-2a \pm \sqrt{64a^2}}{30}$$

$$x = \frac{-2a \pm 8a}{30}$$

$$x = \frac{-2a + 8a}{30} \wedge x = \frac{-2a - 8a}{30}$$

$$x = \frac{6a}{30} \wedge x = \frac{-10a}{30}$$

$$x = \frac{a}{5} \wedge x = \frac{-a}{3}$$

$$S.S = \left\{\frac{-a}{3}, \frac{a}{5}\right\}.$$

(19) $(x-a)(x-b) + (x-b)(x-c) + (x-c)(x-a) = 0$

$$\{x^2 - ax - bx + ab\} + \{x^2 - bx - cx + bc\} + \{x^2 - cx - ax + ac\} = 0$$

$$x^2 - ax - bx + ab + x^2 - bx - cx + bc + x^2 - cx - ax + ac = 0$$

$$3x^2 - 2ax - 2bx - 2cx + ab + bc + ca = 0$$

$$3x^2 - 2(a+b+c)x + (ab+bc+ca) = 0$$

Comparing with $Ax^2 + Bx + C = 0$

$A = 3, \quad B = -2(a+b+c), \quad C = (ab+bc+ca)$

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

$$x = \frac{2(a+b+c) \pm \sqrt{4(a+b+c)^2 - 4(3)(ab+bc+ca)}}{2(3)}$$

$$x = \frac{2(a+b+c) \pm 2\sqrt{(a+b+c)^2 - 3(ab+bc+ca)}}{2(3)}$$

$$x = \cancel{2}\left\{\frac{(a+b+c) \pm \sqrt{(a^2+b^2+c^2+2ab+2bc+2c-3ab-3bc-3ca)}}{\cancel{2} \cdot 3}\right.$$

$$x = \frac{(a+b+c) \pm \sqrt{a^2+b^2+c^2-ab-bc-ca}}{3}$$

which is Solution of Equation.

(20) $(a+b)x^2+(a+2b+c)x+(b+c)=0$

Comparing with $Ax^2+Bx+C=0$

$A=(a+b) \quad B=(a+2b+c) \quad C=(b+c)$

Using Quadratic formula.

$$x=\frac{-B \pm \sqrt{B^2-4AC}}{2A}$$

$$x=\frac{-(a+2b+c) \pm \sqrt{(a+2b+c)^2-4(a+b)(b+c)}}{2(a+b)}$$

$$x=\frac{-(a+2b+c) \pm \sqrt{a^2+\cancel{4b^2}+c^2+\cancel{4ab}+\cancel{4bc}+2ca-\cancel{4ab}-4ac-\cancel{4b^2}-\cancel{4bc}}}{2(a+b)}$$

$$x=\frac{-(a+2b+c) \pm \sqrt{a^2+c^2-2ac}}{2(a+b)}$$

$$x=\frac{-(a+2b+c) \pm \sqrt{(a-c)^2}}{2(a+b)}$$

$$x=\frac{-(a+2b+c) \pm (a-c)}{2(a+b)}$$

$$x=\frac{-(a+2b+c)+(a-c)}{2(a+b)} \wedge x=\frac{-(a+2b+c)-(a-c)}{2(a+b)}$$

$$x=\frac{-\cancel{a}-2b-c+\cancel{a}-c}{2(a+b)} \wedge x=\frac{-a-2b-\cancel{c}-a+\cancel{c}}{2(a+b)}$$

$$x=\frac{-2(b+c)}{2(a+b)} \wedge x=\frac{-2(a+b)}{2(a+b)}$$

$$x=\frac{-(b+c)}{(a+b)} \wedge x=-1$$

$$S.S=\left\{-1, \frac{-(b+c)}{(a+b)}\right\}$$

Equations reducible to Simple Quadratic Equations: (6)

Some times, we are to deal with Complicated Equations. Using some useful Substitutions, we can convert these Equations in Simple (Standard) Quadratic Equations and then can solve with our own choice.

These equations will be appear in different types, we will solve these equation according to their required and suitable substitutions.

Definations:

* The equation in which variable appears as an exponent (Power or index) is called Exponential Equation.

* The equation which remains unchanged when variable is changed by its reciprocal is called Reciprocal Equation.