(iii) $\begin{bmatrix} 3 & -1 & 3 & 0 & -1 \\ 1 & 2 & -1 & -3 & -2 \\ 2 & 3 & 4 & 2 & 5 \\ 2 & 5 & -2 & -2 & 3 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & -1 & -3 & -2 \\ 3 & -1 & 3 & 0 & -1 \\ 2 & 3 & 4 & 2 & 5 \\ 2 & 5 & -2 & -2 & 3 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$R \begin{bmatrix} 1 & 2 & -1 & -3 & -2 \\ 0 & -7 & 6 & 9 & 5 \\ 0 & -1 & 6 & 8 & 9 \\ 0 & 1 & 0 & 4 & 7 \end{bmatrix} \quad \text{By } R_2-3R_1,\ R_3-2R_1,\ R_4-2R_1$$

$$R \begin{bmatrix} 1 & 2 & -1 & -3 & -2 \\ 0 & 1 & 0 & 4 & 7 \\ 0 & -1 & 6 & 8 & 9 \\ 0 & -7 & 6 & 9 & 5 \end{bmatrix} \quad R_4 \leftrightarrow R_2$$

$$R \begin{bmatrix} 1 & 2 & -1 & -3 & -2 \\ 0 & 1 & 0 & 4 & 7 \\ 0 & 0 & 6 & 14 & 16 \\ 0 & 0 & 6 & 37 & 68 \end{bmatrix} \quad \text{By } R_3+R_2,\ R_4+7R_2$$

$$R \begin{bmatrix} 1 & 2 & -1 & -3 & -2 \\ 0 & 1 & 0 & 4 & 7 \\ 0 & 0 & 1 & 7/3 & 8/3 \\ 0 & 0 & 1 & 37/6 & 34/3 \end{bmatrix} \quad \text{By } \tfrac{1}{6}R_3,\ \tfrac{1}{6}R_4$$

$$R \begin{bmatrix} 1 & 2 & -1 & -3 & -2 \\ 0 & 1 & 0 & 4 & 7 \\ 0 & 0 & 1 & 7/3 & 8/3 \\ 0 & 0 & 0 & \frac{23}{6} & \frac{26}{3} \end{bmatrix} \quad \text{By } R_4-R_3$$

$$R \begin{bmatrix} 1 & 2 & -1 & -3 & -2 \\ 0 & 1 & 0 & 4 & 7 \\ 0 & 0 & 1 & 7/3 & 8/3 \\ 0 & 0 & 0 & 1 & 52/23 \end{bmatrix} \quad \text{By } \tfrac{6}{23}R_4$$

This is the echelon form of the matrix and 4 nonzero rows so rank is 4

Exercise:3.5

Q.1 Solve Using Cramer's Rule.

(i) $2x+2y+z=3$
$3x-2y-2z=1$
$5x+y-3z=2$

In Matrix form

$$\begin{bmatrix} 2 & 2 & 1 \\ 3 & -2 & -2 \\ 5 & 1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

Let $A = \begin{pmatrix} 2 & 2 & 1 \\ 3 & -2 & -2 \\ 5 & 1 & -3 \end{pmatrix}$

$|A| = 2\begin{vmatrix} -2 & -2 \\ 1 & -3 \end{vmatrix} - 2\begin{vmatrix} 3 & -2 \\ 5 & -3 \end{vmatrix} + 1\begin{vmatrix} 3 & -2 \\ 5 & 1 \end{vmatrix}$

$|A| = 2(6+2) - 2(-9+10) + 1(3+10)$

$|A| = 2(8) - 2(1) + 1(13)$

$|A| = 16 - 2 + 13 = 27$

Using Cramer's Rule

$$x = \frac{\begin{vmatrix} 3 & 2 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & -3 \end{vmatrix}}{|A|} = \frac{3\begin{vmatrix} -2 & -2 \\ 1 & -3 \end{vmatrix} - 1\begin{vmatrix} 2 & 1 \\ 1 & -3 \end{vmatrix} + 2\begin{vmatrix} 2 & 1 \\ -2 & -2 \end{vmatrix}}{27}$$

$$x = \frac{3(6+2)-1(-6-1)+2(-4+2)}{27}$$

$$x = \frac{24+7-4}{27} = \frac{27}{27} = 1$$

$$y = \frac{\begin{vmatrix} 2 & 3 & 1 \\ 3 & 1 & -2 \\ 5 & 2 & -3 \end{vmatrix}}{|A|} = \frac{2\begin{vmatrix} 1 & -2 \\ 2 & -3 \end{vmatrix} - 3\begin{vmatrix} 3 & 1 \\ 2 & -3 \end{vmatrix} + 5\begin{vmatrix} 3 & 1 \\ 1 & -2 \end{vmatrix}}{27}$$

$$y = \frac{2(-3+4)-3(-9-2)+5(-6-1)}{27}$$

$$y = \frac{2+33-35}{27} = \frac{0}{27} = 0$$

$$z = \frac{\begin{vmatrix} 2 & 2 & 3 \\ 3 & -2 & 1 \\ 5 & 1 & 2 \end{vmatrix}}{|A|} = \frac{2\begin{vmatrix} -2 & 1 \\ 1 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} + 5\begin{vmatrix} 2 & 3 \\ -2 & 1 \end{vmatrix}}{27}$$

$$z = \frac{2(-4-1)-3(4-3)+5(2+6)}{27} = \frac{-10-3+40}{27}$$

$$z = \frac{27}{27} = 1$$

Hence $(x,y,z) = (1,0,1)$

(Similarly (ii) and (iii) Part do yourself)

Q.2 Use matrices to solve the System:

(i) $\begin{cases} x-2y+z=-1 \\ 3x+y-2z=4 \\ \quad y-z=1 \end{cases}$

In matrix form

$$\begin{bmatrix} 1 & -2 & 1 \\ 3 & 1 & -2 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$$

let $A=\begin{bmatrix} 1 & -2 & 1 \\ 3 & 1 & 2 \\ 0 & 1 & -1 \end{bmatrix}$, $B=\begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$, $X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$\because AX=B \Rightarrow X=A^{-1}B$

Now $|A| = \begin{vmatrix} 1 & -2 & 1 \\ 3 & 1 & -2 \\ 0 & 1 & -1 \end{vmatrix}$

$= 1(-1+2)+2(-3-0)+1(3-0)$

$= 1-6+3 = -2 \neq 0$

Cofactor matrix of $A = \begin{bmatrix} \begin{vmatrix} 1 & -2 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} 3 & -2 \\ 0 & -1 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ 0 & 1 \end{vmatrix} \\ -\begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} \\ \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} \end{bmatrix}$

$$= \begin{bmatrix} 1 & 3 & 3 \\ -1 & -1 & -1 \\ 3 & 5 & 7 \end{bmatrix}$$

$\text{Adj}\, A = [\text{Cofactor's Matrix of } A]^t$

$$= \begin{bmatrix} 1 & -1 & 3 \\ 3 & -1 & 5 \\ 3 & -1 & 7 \end{bmatrix}$$

$$A^{-1} = \frac{\text{Adj}A}{|A|} = \frac{-1}{2}\begin{bmatrix} 1 & -1 & 3 \\ 3 & -1 & 5 \\ 3 & -1 & 7 \end{bmatrix}$$

$X = A^{-1}B$

$$X = \frac{-1}{2}\begin{bmatrix} 1 & -1 & 3 \\ 3 & -1 & 5 \\ 3 & -1 & 7 \end{bmatrix}\begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$$

$$= \frac{-1}{2}\begin{bmatrix} -1-4+3 \\ -3-4+5 \\ -3-4+7 \end{bmatrix} = \frac{-1}{2}\begin{bmatrix} -2 \\ -2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} +1 \\ 1 \\ 0 \end{bmatrix}$$

Hence $x=1$, $y=1$, $z=0$

S. S $=\{(1,1,0)\}$ Ans.

similarly try (ii) and (iii) Parts.

Q.3 Use Echelon or reduced Echelon form to Solve the System:

(i) $\begin{cases} x_1-2x_2-2x_3=-1 \\ 2x_1+3x_2+x_3=1 \\ 5x_1-4x_2-3x_3=1 \end{cases}$

Thus Augmented Matrix is:

$$A_b = \left[\begin{array}{ccc|c} 1 & -2 & -2 & -1 \\ 2 & 3 & 1 & 1 \\ 5 & -4 & -3 & 1 \end{array}\right]$$

$$\underset{R}{\sim} \left[\begin{array}{ccc|c} 1 & -2 & -2 & -1 \\ 0 & 7 & 5 & 3 \\ 0 & 6 & 7 & 6 \end{array}\right] \begin{array}{l} \text{By } R_2-2R_1 \\ R_3-5R_1 \end{array}$$

$$\underset{R}{\sim} \left[\begin{array}{ccc|c} 1 & -2 & -2 & -1 \\ 0 & 1 & -2 & -3 \\ 0 & 6 & 7 & 6 \end{array}\right] \begin{array}{l} \text{By} \\ R_2-R_3 \end{array}$$

$$\underset{R}{\sim} \left[\begin{array}{ccc|c} 1 & -2 & -2 & -1 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 19 & 24 \end{array}\right] \begin{array}{l} \text{By} \\ R_3-6R_2 \end{array}$$

$$\underset{R}{\sim} \left[\begin{array}{ccc|c} 1 & -2 & -2 & -1 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 24/19 \end{array}\right] \text{By } \frac{1}{19}R_3$$

$$\overset{R}{\sim}\left[\begin{array}{ccc|c} 1 & -2 & 0 & 29/19 \\ 0 & 1 & 0 & -9/19 \\ 0 & 0 & 1 & 24/19 \end{array}\right] \quad \text{By } R_1+2R_3,\ R_2+2R_3$$

$$\overset{R}{\sim}\left[\begin{array}{ccc|c} 1 & 0 & 0 & 11/19 \\ 0 & 1 & 0 & -9/19 \\ 0 & 0 & 1 & 24/19 \end{array}\right] \quad \text{By } R_1+2R_2$$

So $x_1 = 11/19,\ x_2 = -9/19,\ x_3 = 24/19$

$S.S = \{(11/19, -9/19, 24/19)\}$ Ans.

Similarly (ii) and (iii) try yourself.

Q.4 Solve the homogeneous System:

(i) $\begin{cases} x+2y-2z=0 \\ 2x+y+5z=0 \\ 5x+4y+8z=0 \end{cases}$

The Augmented Matrix is:

$$A_b = \left[\begin{array}{ccc|c} 1 & 2 & -2 & 0 \\ 2 & 1 & 5 & 0 \\ 5 & 4 & 8 & 0 \end{array}\right]$$

$$\overset{R}{\sim}\left[\begin{array}{ccc|c} 1 & 2 & -2 & 0 \\ 0 & -3 & 9 & 0 \\ 0 & -6 & 18 & 0 \end{array}\right] \quad \text{By } R_2-2R_1,\ R_3-5R_1$$

$$\overset{R}{\sim}\left[\begin{array}{ccc|c} 1 & 2 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & -6 & 18 & 0 \end{array}\right] \quad \text{By } \frac{-1}{3}R_2$$

$$\overset{R}{\sim}\left[\begin{array}{ccc|c} 1 & 2 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \text{By } R_3+6R_2$$

$\Rightarrow$ System has infinite Solutions

as rank A_b = Rank A < No. of variables

let $z=t$

So $y-3z=0 \Rightarrow y=3z$

$y=3t$.

Now $x+2y-2z=0$

$x=-2y+2z$

$x=-2(3t)+2(t)$

$x=-6t+2t$

$x=-4t$

Thus $S.S = \{(-4t, 3t, t)\}$

Similarly (ii) and (iii) try yourself.

Q.5 Find value of λ for which System has non-trivial Solution. Also find the Solutions:

(i) $\begin{cases} x+y+z=0 \\ 2x+y-\lambda z=0 \\ x+2y-2z=0 \end{cases}$

The System will have non-trivial Solution if det. of Coefficient matrix $=0$

$$\Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -\lambda \\ 1 & 2 & -2 \end{vmatrix} = 0$$

$$1\begin{vmatrix} 1 & -\lambda \\ 2 & -2 \end{vmatrix} - 1\begin{vmatrix} 2 & -\lambda \\ 1 & -2 \end{vmatrix} + 1\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 0$$

$1(-2+2\lambda)-1(-4+\lambda)+1(4-1)=0$

$2\lambda-2-\lambda+4+3=0$

$\lambda+5=0 \Rightarrow \boxed{\lambda=-5}$

For $\lambda=-5$

$\begin{cases} x+y+z=0 \\ 2x+y+5z=0 \\ x+2y-2z=0 \end{cases}$

Augmented Matrix is: (45)

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 1 & 5 & 0 \\ 1 & 2 & -2 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right] \text{By } R_2-2R_1,\ R_3-R_1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{By } R_3+R_2$$

The System has infinite Solutions as

Rank(A) = Rank(A_b) < No. of Variables

let $z = t$

So $-y+3z=0 \Rightarrow y = 3z$

$y = 3t$

and $x+y+z=0$

$x = -y-z$

$x = -3t-t$

$x = -4t$

Thus $S.S = \{(-4t, 3t, t)\}$.

(ii) $$\begin{cases} x_1+4x_2+\lambda x_3 = 0 \\ 2x_1+x_2-3x_3 = 0 \\ 3x_1+\lambda x_2-4x_3 = 0 \end{cases}$$

The System will have non-trivial Solution if det. of Coeff. matrix $=0$

$$\begin{vmatrix} 1 & 4 & \lambda \\ 2 & 1 & -3 \\ 3 & \lambda & -4 \end{vmatrix} = 0$$

$$1\begin{vmatrix} 1 & -3 \\ \lambda & -4 \end{vmatrix} - 4\begin{vmatrix} 2 & -3 \\ 3 & -4 \end{vmatrix} + \lambda\begin{vmatrix} 2 & 1 \\ 3 & \lambda \end{vmatrix} = 0$$

$$1(-4+3\lambda)-4(-8+9)+\lambda(2\lambda-3)=0$$

$$\cancel{3\lambda}-4-4+2\lambda^2-\cancel{3\lambda}=0$$

$$2\lambda^2 = 8 \Rightarrow \lambda^2 = 4$$

$$\Rightarrow \boxed{\lambda = \pm 2}$$

For $\lambda = 2$

$$\begin{cases} x_1+4x_2+2x_3=0 \\ 2x_1+x_2-3x_3=0 \\ 3x_1+2x_2-4x_3=0 \end{cases}$$

For $\lambda=-2$

$$\begin{cases} x_1+4x_2-2x_3=0 \\ 2x_1+x_2-3x_3=0 \\ 3x_1-2x_2-4x_3=0 \end{cases}$$

Ist by

$$A_b = \left[\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 2 & 1 & -3 & 0 \\ 3 & 2 & -4 & 0 \end{array}\right]$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & -7 & -7 & 0 \\ 0 & -10 & -10 & 0 \end{array}\right] \text{By } R_2-2R_1,\ R_3-3R_1$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -10 & -10 & 0 \end{array}\right] \text{By } -\frac{1}{7}R_2$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{By } R_3+10R_2$$

The System has infinite Solutions as

Rank(A) = Rank(A_b) < No. of Variables.

let $x_3 = t \Rightarrow x_2+x_3=0$

$x_2 = -x_3 = -t$

So $x_1+4x_2+2x_3=0$

$x_1-4t+2t=0 \Rightarrow x_1 = 2t$

Thus $S.S = \{(2t, -t, t)\}$.

For $\lambda = -2$

$$A_b = \left[\begin{array}{ccc|c} 1 & 4 & -2 & 0 \\ 2 & 1 & -3 & 0 \\ 3 & -2 & -4 & 0 \end{array}\right]$$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & 4 & -2 & 0 \\ 0 & -7 & 1 & 0 \\ 0 & -14 & 2 & 0 \end{array}\right] \quad \text{By } R_2 - 2R_1, \; R_3 - 3R_1$$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & 4 & -2 & 0 \\ 0 & 1 & -1/7 & 0 \\ 0 & -14 & 2 & 0 \end{array}\right] \quad \text{By } \tfrac{-1}{7}R_2$$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & 4 & -2 & 0 \\ 0 & 1 & -1/7 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \text{By } R_3 + 14R_2$$

The system has infinite solutions as $Rank(A) = Rank(A_b) <$ No. of variables.

let $x_3 = t$

$\Rightarrow \quad x_2 - \frac{1}{7}x_3 = 0 \Rightarrow x_2 = \frac{1}{7}x_3$

$x_2 = \frac{1}{7}t$

Now $x_1 + 4x_2 - 2x_3 = 0$

$x_1 = -4(\frac{1}{7}t) + 2(t)$

$x_1 = \frac{14-4}{7}t = \frac{10}{7}t$

Thus $S.S = \{(\frac{10}{7}t, \frac{1}{7}t, t)\}$.

Q.6 Find λ if System has no unique Solution, also Solve.

$$\begin{cases} x_1 + 4x_2 + \lambda x_3 = 2 \\ 2x_1 + x_2 - 2x_3 = 11 \\ 3x_1 + 2x_2 - 2x_3 = 16 \end{cases}$$

(46) System will not has unique Sol if det. of Coeff. matrix $=$

$$\begin{vmatrix} 1 & 4 & \lambda \\ 2 & 1 & -2 \\ 3 & 2 & -2 \end{vmatrix} = 0$$

$\Rightarrow 1(-2+4) - 4(-4+6) + \lambda(4-3) = 0$

$\Rightarrow 2 - 8 + \lambda = 0 \Rightarrow \boxed{\lambda = 6}$

Now for non-unique solution, Sy

$$\begin{cases} x_1 + 4x_2 + 6x_3 = 2 \\ 2x_1 + x_2 - 2x_3 = 11 \\ 3x_1 + 2x_2 - 2x_3 = 16 \end{cases}$$

$$A_b = \left[\begin{array}{ccc|c} 1 & 4 & 6 & 2 \\ 2 & 1 & -2 & 11 \\ 3 & 2 & -2 & 16 \end{array}\right]$$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & 4 & 6 & 2 \\ 0 & -7 & -14 & 7 \\ 0 & -10 & -20 & 10 \end{array}\right] \quad \text{By } R_2 - 2R_1, \; R_3 - 3R_1$$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & 4 & 6 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & -10 & -20 & 10 \end{array}\right] \quad \text{By } \tfrac{-1}{7}R_2$$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & 4 & 6 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_3 + 10R_2$$

Clearly $Rank(A) = Rank(A_b) <$ No. of Variable so System has infinite Solutions.

let $x_3 = t \Rightarrow x_2 + 2x_3 = -1$

$x_2 = -1 - 2t$

$x_1 + 4x_2 + 6x_3 = 2$

$x_1 = 2 - 4(-1-2t) - 6t = 2 + 4 + 8t - 6t$

$x_1 = 6 + 2t$

Thus $S.S = \{(6+2t, -1-2t, t)\}$.

The End.