~~Similarly Elementary Column~~ Operation:-

NOTE:-

"The matrix obtained after Elementary row or Column operation is called Elementary matrix."

Augment Matrix:-

"The matrix obtained by appending (i.e) any row or Column of Constants on the left of matrix or below the matrix is called Augment Matrix."

Upper Triangular Matrix:-

"The matrix in which the element below the principle diagonal are zero's is called Upper Triangular Matrix." $a_{ij}=0$ $i>j$

Lower Triangular Matrix:-

"The matrix in which the elements above the principle diagonal are zero's is called lower Triangular Matrix."

$a_{ij}=0$ for $i<j$

$$A=\begin{bmatrix} 2 & 2 & 11 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix} \qquad B=\begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 13 & 5 & 6 \end{bmatrix}$$

Upper Triangular matrix Lower Triangular Matrix.

Triangular Matrix:-

"The matrix which is either upper triangular or lower triangular is called Triangular Matrix."

"Diagonal matrix is upper aswell as lower triangular matrix."

Conjugate of a Matrix:-

Let $A=[a_{ij}]$ be a matrix consists of complex Entries then the Conjugate is denoted by $\bar{A}$ and is defined as

$\bar{A}=[\bar{a}_{ij}]$

Hermitian Matrix:-

The square matrix A with Complex entries is called Hermitian Matrix if

$(\bar{A})^t = A$

Skew-Hermitian Matrix:-

The square matrix A with Complex Entries is called skew Hermitian Matrix if

$(\bar{A})^t = -A$

Tahir Mahmood.
TAHIR

Exercise: 3.4

Echelon Form of A Matrix:-

An $m\times n$ matrix A is called echelon form if

(1) In consecutive non-zero rows, the number of zero in the next rows are greater and greater

(2) Every non zero row in the matrix precedes every zero row.

(3) The first non-zero entry (or leading entry or Pivotal) in each row is 1.

Reduced Echelon form:-

An $m\times n$ matrix A is said to be reduced echelon form if there are zero's in the echelon form above pivotals (or leading entries).

Rank of a Matrix:

"The number of non-zero rows in any matrix after echelon form gives the rank of any matrix."

Tahir Mahmood. TAHIR
Govt. College
Gujranwala.

Q.1 If $A=\begin{bmatrix}1&-2&5\\-2&3&-1\\5&-1&0\end{bmatrix}$ and $B=\begin{bmatrix}-3&1&-2\\1&0&-1\\-2&-1&2\end{bmatrix}$ then show that $A+B$ is symmetric.

$$A+B=\begin{bmatrix}1&-2&5\\-2&3&-1\\5&-1&0\end{bmatrix}+\begin{bmatrix}-3&1&-2\\1&0&-1\\-2&-1&2\end{bmatrix}$$

$$A+B=\begin{bmatrix}1-3&-2+1&5-2\\-2+1&3+0&-1-1\\5-2&-1-1&0+2\end{bmatrix}$$

$$A+B=\begin{bmatrix}-2&-1&3\\-1&3&-2\\3&-2&2\end{bmatrix}$$

$(A+B)$ will be symmetric if

$$(A+B)^t=(A+B)$$

Now $(A+B)^t=\begin{bmatrix}-2&-1&3\\-1&3&-2\\3&-2&2\end{bmatrix}^t$

$$(A+B)^t=\begin{bmatrix}-2&-1&3\\-1&3&-2\\3&-2&2\end{bmatrix}=(A+B)$$

So $(A+B)$ is symmetric matrix.

Q.2 If $A=\begin{bmatrix}1&2&0\\3&2&-1\\-1&3&2\end{bmatrix}$ then show

(i) $A+A^t$ is symmetric

$$A^t=\begin{bmatrix}1&3&-1\\2&2&3\\0&-1&2\end{bmatrix}$$

Now

$$A+A^t=\begin{bmatrix}1&2&0\\3&2&-1\\-1&3&2\end{bmatrix}+\begin{bmatrix}1&3&-1\\2&2&3\\0&-1&2\end{bmatrix}$$

$$A+A^t=\begin{bmatrix}1+1&2+3&0+(-1)\\3+2&2+2&-1+3\\-1+0&3-1&2+2\end{bmatrix}$$

$$A+A^t=\begin{bmatrix}2&5&-1\\5&4&2\\-1&2&4\end{bmatrix}$$

$$(A+A^t)^t=\begin{bmatrix}2&5&-1\\5&4&2\\-1&2&4\end{bmatrix}$$

$\because (A+A^t)^t=(A+A^t)$

So $A+A^t$ is symmetric matrix.

Tahir Mahmood
M.Sc. (Math)
Mob No:0345-6510779

(ii) $A-A^t$ is skew symmetric

$$A=\begin{pmatrix}1&2&0\\3&2&-1\\-1&3&2\end{pmatrix} \quad A^t=\begin{pmatrix}1&3&-1\\2&2&3\\0&-1&2\end{pmatrix}$$

Now

$$A-A^t=\begin{pmatrix}1&2&0\\3&2&-1\\-1&3&2\end{pmatrix}-\begin{pmatrix}1&3&-1\\2&2&3\\0&-1&2\end{pmatrix}$$

$$A-A^t=\begin{pmatrix}1-1&2-3&0+1\\3-2&2-2&-1-3\\-1-0&3+1&2-2\end{pmatrix}$$

$$A-A^t=\begin{pmatrix}0&-1&1\\1&0&-4\\-1&4&0\end{pmatrix}$$

Now

$$[A-A^t]^t=\begin{pmatrix}0&1&-1\\-1&0&+4\\1&-4&0\end{pmatrix}$$

$$(A-A^t)^t=-\begin{pmatrix}0&-1&1\\1&0&-4\\-1&4&0\end{pmatrix}$$

$$[A-A^t]^t=-(A-A^t)$$

So $A-A^t$ is skew symmetric.

Q.3 Hint Suppose any matrix of 3rd order and do as Q.2

Q.4 If A and B are symmetric and $AB=BA$, show that AB is symmetric.

Proof:- We know that

A and B are symmetric then

$A=A^t$ and $B=B^t$

Also $AB=BA$

We want to show AB is symmetric

i.e. $(AB)^t=AB$.

Now, we know that

$(AB)^t=B^tA^t$

$\because A=A^t$ and $B=B^t$

$\because (AB)^t=BA$

But $AB=BA$

So $(AB)^t=AB$ (Proved)

Q.5 Show that AA^t and A^tA are symmetric for 2×3 order matrix.

Let $A=\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\end{pmatrix}$

$$A^t=\begin{pmatrix}a_{11}&a_{21}\\a_{12}&a_{22}\\a_{13}&a_{23}\end{pmatrix}$$

Now $AA^t=\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\end{pmatrix}\begin{pmatrix}a_{11}&a_{21}\\a_{12}&a_{22}\\a_{13}&a_{23}\end{pmatrix}$

$$AA^t=\begin{pmatrix}a_{11}^2+a_{12}^2+a_{13}^2 & a_{11}a_{21}+a_{12}a_{22}+a_{13}a_{23}\\ a_{11}a_{21}+a_{12}a_{22}+a_{13}a_{23} & a_{21}^2+a_{22}^2+a_{23}^2\end{pmatrix}$$

Now $(AA^t)^t=\begin{pmatrix}a_{11}^2+a_{12}^2+a_{13}^2 & a_{11}a_{21}+a_{12}a_{22}+a_{13}a_{23}\\ a_{11}a_{21}+a_{12}a_{22}+a_{13}a_{23} & a_{21}^2+a_{22}^2+a_{23}^2\end{pmatrix}$

$\because (AA^t)^t=AA^t$

So AA^t is symmetric matrix.

Now $A^tA=\begin{pmatrix}a_{11}&a_{21}\\a_{12}&a_{22}\\a_{13}&a_{23}\end{pmatrix}\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\end{pmatrix}$

$$A^tA=\begin{pmatrix}a_{11}^2+a_{21}^2 & a_{11}a_{12}+a_{21}a_{22} & a_{11}a_{13}+a_{21}a_{23}\\ a_{12}a_{11}+a_{22}a_{21} & a_{12}^2+a_{22}^2 & a_{12}a_{13}+a_{22}a_{23}\\ a_{13}a_{11}+a_{23}a_{21} & a_{13}a_{12}+a_{23}a_{22} & a_{13}^2+a_{23}^3\end{pmatrix}$$

Now $(A^tA)^t=\begin{pmatrix}a_{11}^2+a_{21}^2 & a_{11}a_{12}+a_{21}a_{22} & a_{11}a_{13}+a_{21}a_{23}\\ a_{11}a_{12}+a_{21}a_{22} & a_{11}^2+a_{22}^2 & a_{12}a_{13}+a_{22}a_{23}\\ a_{11}a_{13}+a_{23}a_{23} & a_{13}a_{12}+a_{23}a_{22} & a_{13}^2+a_{23}^2\end{pmatrix}$

$\because (A^tA)^t=A^tA$

So A^tA is symmetric

Q6 If $A=\begin{bmatrix} i & 1+i \\ 1 & -i \end{bmatrix}$, show that

(i) $A+(\bar{A})^t$ is Hermitian

$\because A=\begin{bmatrix} i & 1+i \\ 1 & -i \end{bmatrix} \qquad \bar{A}=\begin{bmatrix} -i & 1-i \\ 1 & i \end{bmatrix}$

Now $\qquad (\bar{A})^t=\begin{bmatrix} -i & 1 \\ 1-i & i \end{bmatrix}$

$$A+(\bar{A})^t=\begin{bmatrix} i & 1+i \\ 1 & -i \end{bmatrix}+\begin{bmatrix} -i & 1 \\ 1-i & i \end{bmatrix}$$

$$A+(\bar{A})^t=\begin{bmatrix} i-i & 1+i+1 \\ 1+1-i & -i+i \end{bmatrix}$$

$$A+(\bar{A})^t=\begin{bmatrix} 0 & 2+i \\ 2-i & 0 \end{bmatrix}$$

$A+(\bar{A})^t$ will be Hermitian if

$$\left(\overline{A+(\bar{A})^t}\right)^t = A+(\bar{A})^t$$

So $\left(\overline{A+(\bar{A})^t}\right)=\begin{bmatrix} 0 & 2-i \\ 2+i & 0 \end{bmatrix}$

$$\left[\overline{A+(\bar{A})^t}\right]^t=\begin{bmatrix} 0 & 2+i \\ 2-i & 0 \end{bmatrix}$$

$$\left(\overline{A+(\bar{A})^t}\right)^t = A+(\bar{A})^t$$

So $A+(\bar{A})^t$ is Hermitian Matrix.

(ii) $A-A^t$ is Skew Hermitian.

Now $A-(\bar{A})^t=\begin{bmatrix} i & 1+i \\ 1 & -i \end{bmatrix}-\begin{bmatrix} -i & 1 \\ 1-i & i \end{bmatrix}$

$$A-(\bar{A})^t=\begin{bmatrix} i+i & 1+i-1 \\ 1-1+i & -i-i \end{bmatrix}$$

$$A-(\bar{A})^t=\begin{bmatrix} 2i & i \\ i & -2i \end{bmatrix}$$

$A-(\bar{A})^t$ will be Skew Hermitian if

$$\left[\overline{A-(\bar{A})^t}\right]^t=-\left(A-(\bar{A})^t\right)$$

So $\left[\overline{A-(\bar{A})^t}\right]=\begin{bmatrix} -2i & -i \\ -i & 2i \end{bmatrix}$

$$\left[\overline{A-(\bar{A})^t}\right]^t=\begin{bmatrix} -2i & -i \\ -i & 2i \end{bmatrix}$$

$$\left[\overline{A-(\bar{A})^t}\right]^t=-\begin{bmatrix} 2i & i \\ i & -2i \end{bmatrix}$$

$$\left[\overline{A-(\bar{A})^t}\right]^t=-\left(A-(\bar{A})^t\right) \quad \text{(Proved).}$$

Q7 If A is Symmetric or Skew Symmetric, Show A^2 is Symmetric.

If A is Symmetric or Skew Symmetric.

then Either $A^t=A$ or $A^t=-A$

A^2 will be Symmetric if $(A^2)^t=A^2$

So $(A^2)^t=(AA)^t$

$(A^2)^t=A^t\cdot A^t$

$(A^2)^t=A\cdot A$ or $(-A)(-A)$

$(A^2)^t=A^2 \Rightarrow (A^2)^t=A^2$

In both the cases

$(A^2)^t=A^2$

So A^2 is Symmetric.

Q8 If $A=\begin{bmatrix} 1 \\ 1+i \\ i \end{bmatrix}$ Find $A(\bar{A})^t$

$$\bar{A}=\begin{bmatrix} 1 \\ 1-i \\ -i \end{bmatrix}$$

$$(\bar{A})^t=\begin{bmatrix} 1 & 1-i & -i \end{bmatrix}$$

$$\because A(\bar{A})^t=\begin{bmatrix} 1 \\ 1+i \\ i \end{bmatrix}\begin{bmatrix} 1 & 1-i & -i \end{bmatrix}$$

$$A(\bar{A})^t=\begin{bmatrix} 1 & 1-i & -i \\ 1+i & 1-i^2 & -i-i^2 \\ i & i-i^2 & -i^2 \end{bmatrix}$$

$$A(\bar{A})^t=\begin{bmatrix} 1 & 1-i & -i \\ 1+i & 1+1 & -i+1 \\ i & i+1 & +1 \end{bmatrix} \quad (\because i^2=-1)$$

$$A(\bar{A})^t=\begin{bmatrix} 1 & 1-i & -i \\ 1+i & 2 & 1-i \\ i & 1+i & 1 \end{bmatrix} \quad \text{Ans.}$$

Tahir Mahmood TAHIR

Q.9 Find the inverse of the following matrices, Also by row and Column operation method.

(i) $\begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2 & -2 & 2 \end{bmatrix}$

Let $A = \begin{pmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2 & -2 & 2 \end{pmatrix}$

Cofactors of A are

$A_{11} = (-1)^{1+1} \begin{vmatrix} -2 & 0 \\ -2 & 2 \end{vmatrix} = 1(-4+0) = -4$

$A_{12} = (-1)^{1+2} \begin{vmatrix} 0 & 0 \\ -2 & 2 \end{vmatrix} = -1(0-0) = 0$

$A_{13} = (-1)^{1+3} \begin{vmatrix} 0 & -2 \\ -2 & -2 \end{vmatrix} = 1(0-4) = -4$

$A_{21} = (-1)^{2+1} \begin{vmatrix} 2 & -3 \\ -2 & 2 \end{vmatrix} = -1(4-6) = 2$

$A_{22} = (-1)^{2+2} \begin{vmatrix} 1 & -3 \\ -2 & 2 \end{vmatrix} = 1(2-6) = -4$

$A_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 2 \\ -2 & -2 \end{vmatrix} = -1(-2+4) = -2$

$A_{31} = (-1)^{3+1} \begin{vmatrix} 2 & -3 \\ -2 & 0 \end{vmatrix} = 1(-6) = -6$

$A_{32} = (-1)^{3+2} \begin{vmatrix} 1 & -3 \\ 0 & 0 \end{vmatrix} = -1(0-0) = 0$

$A_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 2 \\ 0 & -2 \end{vmatrix} = 1(-2-0) = -2$

Matrix of Cofactors of $A = \begin{pmatrix} -4 & 0 & -4 \\ 2 & -4 & -2 \\ -6 & 0 & -2 \end{pmatrix}$

$Adj A = [\text{Cofactor's Matrix of } A]^t$

$Adj A = \begin{pmatrix} -4 & 2 & -6 \\ 0 & -4 & 0 \\ -4 & -2 & -2 \end{pmatrix}$

$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$

$= 1(-4) + 2(0) + (-3)(-4)$

$= -4 + 0 + 12 = 8$

$|A| = 8$

$A^{-1} = \dfrac{Adj A}{|A|}$

$A^{-1} = \dfrac{1}{8}\begin{pmatrix} -4 & 2 & -6 \\ 0 & -4 & 0 \\ -4 & -2 & -2 \end{pmatrix}$

$A^{-1} = \begin{pmatrix} -1/2 & 1/4 & -3/4 \\ 0 & -1/2 & 0 \\ -1/2 & -1/4 & -1/4 \end{pmatrix}$ Ans

Now By Row Operation.

$[A \vdots I] = \left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 1 & 0 \\ -2 & -2 & 2 & 0 & 0 & 1 \end{array}\right]$

By $R_3 + 2R_1$ we get

$\left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 1 & 0 \\ 0 & 2 & -4 & 2 & 0 & 1 \end{array}\right]$

$\frac{-1}{2}R_2$ we get

$\left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/2 & 0 \\ 0 & 2 & -4 & 2 & 0 & 1 \end{array}\right]$

By $R_3 - 2R_2$ we get

$\left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/2 & 0 \\ 0 & 0 & -4 & 2 & 1 & 1 \end{array}\right]$

$\left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/2 & 0 \\ 0 & 0 & 1 & -1/2 & -1/4 & -1/4 \end{array}\right]$ By $\frac{-1}{4}R_3$

Now $R_1 - 2R_2$

$\left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1/2 & 0 \\ 0 & 0 & 1 & -1/2 & -1/4 & -1/4 \end{array}\right]$

Tahir Mahmood
M.Sc. (Math)
Mob No: 0345-6510779

Now By $R_1 + 3R_3$, we get

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{4} & -\frac{3}{4} \\ 0 & 1 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \end{array}\right]$$

So $[I \mid A^{-1}]$

Hence by row Operation

$$A^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{4} & -\frac{3}{4} \\ 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

By Column Operation

$$\left[\begin{array}{ccc} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2 & -2 & 2 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & -2 & 0 \\ -2 & 2 & -4 \\ \hline 1 & -2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \begin{array}{l} \text{By} \\ C_2 - 2C_1, \\ C_3 + 3C_1 \end{array}$$

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & -1 & 1 \\ \hline 1 & 1 & -\frac{3}{4} \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{4} \end{array}\right] \quad \text{By } -\tfrac{1}{2}C_2 \text{ and } -\tfrac{1}{4}C_3$$

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline -\frac{1}{2} & \frac{1}{4} & -\frac{3}{4} \\ 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \end{array}\right] \quad \text{By } C_2 + C_3 \text{ and } C_1 + 2C_3$$

Hence the inverse A is

$$A^{-1} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{4} & -\frac{3}{4} \\ 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \end{pmatrix}$$

"Similarly (ii) and (iii) should be done by the Students"

Q.10 Find the ranks of the matrices.

(i) $\begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & -6 & 5 & 1 \\ 3 & 5 & 4 & -3 \end{bmatrix}$

$$\underset{\sim}{R} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & -4 & 1 & -1 \\ 0 & 8 & -2 & -6 \end{bmatrix} \begin{array}{l} \text{By } R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -\frac{1}{4} & \frac{1}{4} \\ 0 & 1 & -\frac{1}{4} & -\frac{3}{4} \end{bmatrix} \begin{array}{l} \text{By } -\frac{1}{4}R_2 \\ \frac{1}{8}R_3 \end{array}$$

$$\underset{\sim}{R} \begin{bmatrix} & -1 & 2 & 1 \\ 0 & 1 & -\frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & -4 \end{bmatrix} \text{By } R_3 - R_2$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -\frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{By } -\tfrac{1}{4}R_3$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & \frac{7}{4} & \frac{5}{4} \\ 0 & 1 & -\frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{array}{l} \text{By} \\ R_1 + R_2 \end{array}$$

$\because$ There are 3 nonzero rows
Hence the rank of matrix is "3".

(ii) $\begin{bmatrix} 1 & -4 & -7 \\ 2 & -5 & 1 \\ 1 & -2 & 3 \\ 3 & -7 & 4 \end{bmatrix}$

$$\underset{\sim}{R} \begin{bmatrix} 1 & -4 & -7 \\ 0 & 3 & 15 \\ 0 & 2 & 10 \\ 0 & 5 & 25 \end{bmatrix} \begin{array}{l} \text{By } R_2 - 2R_1 \\ R_3 - R_1 \\ R_4 - 3R_1 \end{array}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & -4 & -7 \\ 0 & 1 & 5 \\ 0 & 1 & 5 \\ 0 & 1 & 5 \end{bmatrix} \begin{array}{l} \text{By } \frac{R_2}{3} \\ \frac{1}{2}R_3 \\ \frac{1}{5}R_4 \end{array}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & -4 & -7 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \text{By } R_3 - R_2 \\ R_4 - R_2 \end{array}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & 13 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{By } R_1 + 4R_2$$

$\because$ There are 2 non zero rows so
rank of matrix is 2

(iii) $\begin{bmatrix} 3 & -1 & 3 & 0 & -1 \\ 1 & 2 & -1 & -3 & -2 \\ 2 & 3 & 4 & 2 & 5 \\ 2 & 5 & -2 & -2 & 3 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 & -1 & -3 & -2 \\ 3 & -1 & 3 & 0 & -1 \\ 2 & 3 & 4 & 2 & 5 \\ 2 & 5 & -2 & -2 & 3 \end{bmatrix}$ $R_1 \leftrightarrow R_2$

$R \sim \begin{bmatrix} 1 & 2 & -1 & -3 & -2 \\ 0 & -7 & 6 & 9 & 5 \\ 0 & -1 & 6 & 8 & 9 \\ 0 & 1 & 0 & 4 & 7 \end{bmatrix}$ By R_2-3R_1, R_3-2R_1, R_4-2R_1

$R \sim \begin{bmatrix} 1 & 2 & -1 & -3 & -2 \\ 0 & 1 & 0 & 4 & 7 \\ 0 & -1 & 6 & 8 & 9 \\ 0 & -7 & 6 & 9 & 5 \end{bmatrix}$ $R_4 \leftrightarrow R_2$

$R \sim \begin{bmatrix} 1 & 2 & -1 & -3 & -2 \\ 0 & 1 & 0 & 4 & 7 \\ 0 & 0 & 6 & 14 & 16 \\ 0 & 0 & 6 & 37 & 68 \end{bmatrix}$ By R_3+R_2, R_4+7R_2

$R \sim \begin{bmatrix} 1 & 2 & -1 & -3 & -2 \\ 0 & 1 & 0 & 4 & 7 \\ 0 & 0 & 1 & 7/3 & 8/3 \\ 0 & 0 & 1 & 37/6 & 34/3 \end{bmatrix}$ By $\frac{1}{6}R_3$, $\frac{1}{6}R_4$

$R \sim \begin{bmatrix} 1 & 2 & -1 & -3 & -2 \\ 0 & 1 & 0 & 4 & 7 \\ 0 & 0 & 1 & 7/3 & 8/3 \\ 0 & 0 & 0 & \frac{23}{6} & \frac{26}{3} \end{bmatrix}$ By R_4-R_3

$R \sim \begin{bmatrix} 1 & 2 & -1 & -3 & -2 \\ 0 & 1 & 0 & 4 & 7 \\ 0 & 0 & 1 & 7/3 & 8/3 \\ 0 & 0 & 0 & 1 & 52/23 \end{bmatrix}$ By $\frac{6}{23}R_4$

This is the echelon form of the matrix and 4 nonzero rows so rank is 4

Exercise 3.5

Q.1 Solve Using Cramer's Rule.

(i) $2x+2y+z=3$
$3x-2y-2z=1$
$5x+y-3z=2$

In Matrix form

$\begin{bmatrix} 2 & 2 & 1 \\ 3 & -2 & -2 \\ 5 & 1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$

Let $A = \begin{pmatrix} 2 & 2 & 1 \\ 3 & -2 & -2 \\ 5 & 1 & -3 \end{pmatrix}$

$|A| = 2\begin{vmatrix} -2 & -2 \\ 1 & -3 \end{vmatrix} - 2\begin{vmatrix} 3 & -2 \\ 5 & -3 \end{vmatrix} + 1\begin{vmatrix} 3 & -2 \\ 5 & 1 \end{vmatrix}$

$|A| = 2(6+2) - 2(-9+10) + 1(3+10)$

$|A| = 2(8) - 2(1) + 1(13)$

$|A| = 16 - 2 + 13 = 27$

Using Cramer's Rule

$x = \dfrac{\begin{vmatrix} 3 & 2 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & -3 \end{vmatrix}}{|A|} = \dfrac{3\begin{vmatrix} -2 & -2 \\ 1 & -3 \end{vmatrix} - 1\begin{vmatrix} 2 & 1 \\ 1 & -3 \end{vmatrix} + 2\begin{vmatrix} 2 & 1 \\ -2 & -2 \end{vmatrix}}{27}$

$x = \dfrac{3(6+2) - 1(-6-1) + 2(-4+2)}{27}$

$x = \dfrac{24+7-4}{27} = \dfrac{27}{27} = 1$

$y = \dfrac{\begin{vmatrix} 2 & 3 & 1 \\ 3 & 1 & -2 \\ 5 & 2 & -3 \end{vmatrix}}{|A|} = \dfrac{2\begin{vmatrix} 1 & -2 \\ 2 & -3 \end{vmatrix} - 3\begin{vmatrix} 3 & 1 \\ 2 & -3 \end{vmatrix} + 5\begin{vmatrix} 3 & 1 \\ 1 & -2 \end{vmatrix}}{27}$

$y = \dfrac{2(-3+4) - 3(-9-2) + 5(-6-1)}{27}$

$y = \dfrac{2+33-35}{27} = \dfrac{0}{27} = 0$

$z = \dfrac{\begin{vmatrix} 2 & 2 & 3 \\ 3 & -2 & 1 \\ 5 & 1 & 2 \end{vmatrix}}{|A|} = \dfrac{2\begin{vmatrix} -2 & 1 \\ 1 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} + 5\begin{vmatrix} 2 & 3 \\ -2 & 1 \end{vmatrix}}{27}$

$z = \dfrac{2(-4-1) - 3(4-3) + 5(2+6)}{27} = \dfrac{-10-3+40}{27}$

$z = \dfrac{27}{27} = 1$

Hence $(x, y, z) = (1, 0, 1)$

(Similarly (ii) and (iii) Part do yourself)