Q.9 Solve the matrix Equations for A:

(i) $\begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix} A - \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -1 & -4 \\ 3 & 6 \end{bmatrix}$

$= \begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix} A = \begin{bmatrix} -1 & -4 \\ 3 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$

$\begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix} A = \begin{bmatrix} -1+2 & -4+3 \\ 3+(-1) & 6+(-2) \end{bmatrix}$

$\begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix} A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$

Let $B = \begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$

$\because \quad BA = C \Rightarrow B^{-1}BA = B^{-1}C$

$IA = B^{-1}C \Rightarrow A = B^{-1}C$

$|B| = \begin{vmatrix} 4 & 3 \\ 2 & 2 \end{vmatrix} = 8-6 = 2$

$\text{Adj}\,B = \begin{bmatrix} 2 & -3 \\ -2 & 4 \end{bmatrix}$

$B^{-1} = \dfrac{\text{Adj}B}{|B|} = \dfrac{1}{2}\begin{bmatrix} 2 & -3 \\ -2 & 4 \end{bmatrix}$

$\because \quad A = B^{-1}C$

$A = \dfrac{1}{2}\begin{bmatrix} 2 & -3 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$

$A = \dfrac{1}{2}\begin{bmatrix} 2-6 & -2-12 \\ -2+8 & 2+16 \end{bmatrix}$

$A = \dfrac{1}{2}\begin{bmatrix} -4 & -14 \\ 6 & 18 \end{bmatrix}$

$A = \begin{bmatrix} -2 & -7 \\ 3 & 9 \end{bmatrix}$ Ans.

(ii) $A\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} - \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -1 & 5 \end{bmatrix}$

$A\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -1 & 5 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}$

$A\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 2-1 & 0+2 \\ -1+3 & 5+1 \end{bmatrix}$

$A\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix}$

Let $B = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$ $\quad C = \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix}$

$\therefore \quad AB = C \Rightarrow ABB^{-1} = CB^{-1}$

$AI = CB^{-1} \Rightarrow A = CB^{-1}$

$|B| = \begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix} = 6-4 = 2$

$\text{Adj}\,B = \begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix}$

$B^{-1} = \dfrac{\text{Adj}B}{|B|} = \dfrac{1}{2}\begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix}$

$A = CB^{-1}$

$A = \dfrac{1}{2}\begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix}$

$A = \dfrac{1}{2}\begin{bmatrix} 2-8 & -1+6 \\ 4-24 & -2+18 \end{bmatrix}$

$A = \dfrac{1}{2}\begin{bmatrix} -6 & 5 \\ -20 & 16 \end{bmatrix}$

$A = \begin{bmatrix} -3 & 5/2 \\ -10 & 8 \end{bmatrix}$ Ans.

Minor Of an Element:-

"If A is an $m \times n$ matrix then minor of an element is defined as determinant of matrix of $(m-1)\times(n-1)$ order."

This implies that a minor is a matrix having one row and one Column less.

Minor is denoted by M_{ij} and its is the determinant of the matrix M_{ij}.

Cofactor of an Element:-

The Cofactor of an element is denoted by A_{ij} and is defined as

$A_{ij} = (-1)^{i+j} M_{ij}$

where M_{ij} is the minor of row i and Column j.

Determinant of Matrix of order $n \geq 3$

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{23} & a_{33} \end{bmatrix}$

then A determinant is denoted by $|A|$ and is

$$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

also $|A| = a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$

Can also be written as

$$|A| = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$$

or $|A| = a_{11}M_{11} - a_{21}M_{21} + a_{31}M_{31}$

In Simplified form

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

or $|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

Properties of Determinants:

P.1: In any square matrix A

$$|A| = |A^t|$$

Proof:- Consider $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

and $A^t = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$|A| = a_{11}a_{22} - a_{12}a_{21} \quad \text{---} (1)$$

Now $|A^t| = \begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{vmatrix}$

$$|A^t| = a_{11}a_{22} - a_{12}a_{21} \quad \text{---} (2)$$

From (1) and (2)

$$|A| = |A^t| \quad \text{(Proved)}$$

P.2: If any row or Column has zero elements then the determinant will be zero.

Proof:- Consider a matrix such that

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$|A| = a_{11}\begin{vmatrix} 0 & 0 \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} 0 & 0 \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} 0 & 0 \\ a_{31} & a_{32} \end{vmatrix}$$

$$|A| = a_{11}(0-0) - a_{12}(0-0) + a_{13}(0-0)$$

$$|A| = a_{11}(0) - a_{12}(0) + a_{13}(0)$$

$$|A| = 0 + 0 + 0$$

$$|A| = 0 \quad \text{(Proved)}$$

P.3: In any Square matrix, if two rows or columns are identical (Similar) then determinant will be zero.

Proof:- Consider a matrix Such that

$$A = \begin{bmatrix} a_{11} & a_{11} & a_{13} \\ a_{21} & a_{21} & a_{23} \\ a_{31} & a_{31} & a_{33} \end{bmatrix}$$

$$|A| = a_{11}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} - a_{11}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{21} \\ a_{31} & a_{31} \end{vmatrix}$$

$$|A| = a_{11}(a_{21}a_{33} - a_{23}a_{31}) - a_{11}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{31} - a_{21}a_{31})$$

$$= \cancel{a_{11}a_{21}a_{33}} - \cancel{a_{11}a_{23}a_{31}} - \cancel{a_{11}a_{21}a_{33}} + \cancel{a_{11}a_{23}a_{31}} + a_{13}(0)$$

$$= 0 - 0 + 0$$

$$= 0 \quad \text{Hence (proved).}$$

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P-4: Determinant of a diagonal matrix is the product of the elements of the principal diagonal.

Proof:- Consider a matrix such that

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

then

$$|A| = a_{11}\begin{vmatrix} a_{22} & 0 \\ 0 & a_{33} \end{vmatrix} - 0\begin{vmatrix} 0 & 0 \\ 0 & a_{33} \end{vmatrix} + 0\begin{vmatrix} 0 & a_{22} \\ 0 & 0 \end{vmatrix}$$

$$|A| = a_{11}(a_{22}a_{33} - 0) - 0(0-0) + 0(0-0)$$

$$|A| = a_{11}a_{22}a_{33} - 0 + 0 + 0$$

$$|A| = a_{11}a_{22}a_{33} \quad \text{(Proved)}$$

P-5: In any square matrix, if the two rows or Columns are interchange then the determinant will be the -ve times the determinant of the matrix.

Proof Consider a matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$|A| = a_{11}a_{22} - a_{12}a_{21} \quad \text{—— (1)}$$

Now by Changeing two rows

$$A' = \begin{bmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{bmatrix}$$

$$|A'| = \begin{vmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{vmatrix}$$

$$|A'| = a_{21}a_{12} - a_{11}a_{22}$$

$$|A'| = -(a_{11}a_{22} - a_{12}a_{21}) \quad \text{From (1)}$$

$$|A'| = -|A| \quad \text{(Proved)}$$

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P6: If the entries of row or Column is multiplied by any number $K \in \mathbb{R}$ then the determinant will be $K|A|$.

Proof:- let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$$\because |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$|A| = a_{11}a_{22} - a_{12}a_{21}$$

Now $\begin{bmatrix} Ka_{11} & a_{12} \\ Ka_{21} & a_{22} \end{bmatrix}$

$$\begin{vmatrix} Ka_{11} & a_{12} \\ Ka_{21} & a_{22} \end{vmatrix} = Ka_{11}a_{22} - Ka_{12}a_{21}$$

$$= K(a_{11}a_{22} - a_{12}a_{21})$$

$$= K|A| \quad \text{(Proved)}$$

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P-7: If any row or Column of a determinant Consist of two term in addition or Subtraction, then the resulting determinant will be the sum or difference of the two determinants as

$$\begin{vmatrix} a_{11}+b_{11} & a_{12} \\ a_{21}+b_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} b_{11} & a_{12} \\ b_{21} & a_{22} \end{vmatrix}$$

Proof:- The determinant of LHS is

$$= a_{11}+b_{11}(a_{22}) - a_{12}(a_{21}+b_{21})$$

$$= a_{11}a_{22} + b_{11}a_{22} - a_{12}a_{21} - a_{12}b_{21} \quad \text{—— (1)}$$

The determinant of RHS is

$$(a_{11}a_{22} - a_{12}a_{21}) + (b_{11}a_{22} - a_{12}b_{21})$$

$$a_{11}a_{22} + b_{11}a_{22} - a_{21}a_{12} - a_{12}b_{21} \quad \text{—— (2)}$$

LHS = RHS (Proved)

Exercise:33

Q.1 Evaluate the following determinants:

(i) $\begin{vmatrix} 5 & -2 & -4 \\ 3 & -1 & -3 \\ -2 & 1 & 2 \end{vmatrix}$

$= 5\begin{vmatrix} -1 & -3 \\ 1 & 2 \end{vmatrix} + 2\begin{vmatrix} 3 & -3 \\ -2 & 2 \end{vmatrix} - 4\begin{vmatrix} 3 & -1 \\ -2 & 1 \end{vmatrix}$

$= 5(-2+3) + 2(6-6) - 4(3-2)$

$= 5(1) + 2(0) - 4(1) = 5 - 4 = 1$ Ans.

(ii) $\begin{vmatrix} 5 & 2 & -3 \\ 3 & -1 & 1 \\ -2 & 1 & -2 \end{vmatrix}$

$= 5\begin{vmatrix} -1 & 1 \\ 1 & -2 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 \\ -2 & -2 \end{vmatrix} - 3\begin{vmatrix} 3 & -1 \\ -2 & 1 \end{vmatrix}$

$= 5(2-1) - 2(-6+2) - 3(3-2)$

$= 5(1) - 2(-4) - 3(1)$

$= 5 + 8 - 3 = 10$ Ans.

(iii) $\begin{vmatrix} 1 & 2 & -3 \\ -1 & 3 & 4 \\ -2 & 5 & 6 \end{vmatrix}$

$= 1\begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix} - 2\begin{vmatrix} -1 & 4 \\ -2 & 6 \end{vmatrix} - 3\begin{vmatrix} -1 & 3 \\ -2 & 5 \end{vmatrix}$

$= 1(18-20) - 2(-6+8) - 3(-5+6)$

$= 1(-2) - 2(2) - 3(1)$

$= -2 - 4 - 3 = -9$

(iv) $\begin{vmatrix} a+l & a-l & a \\ a & a+l & a-l \\ a-l & a & a+l \end{vmatrix}$

$= a+l\begin{vmatrix} a+l & a-l \\ a & a+l \end{vmatrix} - (a-l)\begin{vmatrix} a & a-l \\ a-l & a+l \end{vmatrix} + a\begin{vmatrix} a & a+l \\ a-l & a \end{vmatrix}$

$= (a+l)(\cancel{a^2} + l^2 + 2al - \cancel{a^2} + al) - (a-l)(\cancel{a^2} + al - \cancel{a^2} - l^2 + 2al)$

$+ a(\cancel{a^2} - \cancel{a^2} + l^2)$

$= (a+l)(l^2 + 3al) - (a-l)(3al - l^2)$

$+ al^2$

$= al^2 + 3a^2l + l^3 + 3al^2 - (3a^2l - al^2 - 3al^2 + l^3)$

$+ al^2$

$= 4al^2 + \cancel{3a^2l} + \cancel{l^3} - \cancel{3a^2l} + 4al^2 - \cancel{l^3} + al^2$

$= 4al^2 + 4al^2 + al^2$

$= 9al^2$ Ans.

(v) $\begin{vmatrix} 1 & 2 & -2 \\ -1 & 1 & -3 \\ 2 & 4 & -1 \end{vmatrix}$

$= 1\begin{vmatrix} 1 & -3 \\ 4 & -1 \end{vmatrix} - 2\begin{vmatrix} -1 & -3 \\ 2 & -1 \end{vmatrix} - 2\begin{vmatrix} -1 & 1 \\ 2 & 4 \end{vmatrix}$

$= 1(-1+12) - 2(1+6) - 2(-4-2)$

$= 1(11) - 2(7) - 2(-6)$

$= 11 - 14 + 12 = 9$ Ans.

(vi) $\begin{vmatrix} 2a & a & a \\ b & 2b & b \\ c & c & 2c \end{vmatrix}$

$= 2a\begin{vmatrix} 2b & b \\ c & 2c \end{vmatrix} - a\begin{vmatrix} b & b \\ c & 2c \end{vmatrix} + a\begin{vmatrix} b & 2b \\ c & c \end{vmatrix}$

$= 2a(4bc - bc) - a(2bc - bc) + a(bc - 2bc)$

$= 2a(3bc) - a(bc) + a(-bc)$

$= 6abc - abc - abc$

$= 4abc$ Ans.

Q.2 Without Expansion Show that:

(i) $\begin{vmatrix} 6 & 7 & 8 \\ 3 & 4 & 5 \\ 2 & 3 & 4 \end{vmatrix} = 0$

$LHS = \begin{vmatrix} 6 & 7 & 8 \\ 3 & 4 & 5 \\ 2 & 3 & 4 \end{vmatrix}$

By $C_2 - \left(\frac{C_1 + C_2}{2}\right)$, we get

$\begin{vmatrix} 6 & 7-7 & 8 \\ 3 & 4-4 & 5 \\ 2 & 3-3 & 4 \end{vmatrix} = \begin{vmatrix} 6 & 0 & 8 \\ 3 & 0 & 5 \\ 2 & 0 & 4 \end{vmatrix} = 0$

$\because$ One Column is zero.

(ii) $\begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{vmatrix} = 0$

$LHS = \begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{vmatrix}$

By $(C_2 - C_1) + C_3$

$= \begin{vmatrix} 2 & 3 & -1+1 \\ 1 & 1 & 0+0 \\ 2 & -3 & 5+(-5) \end{vmatrix}$

$= \begin{vmatrix} 2 & 3 & 0 \\ 1 & 1 & 0 \\ 2 & -3 & 0 \end{vmatrix} = 0$

$\because C_3$ is Consists of zeros.

(iii) $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0$

$LHS = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$

By $(R_1 + R_3) - R_2$

$= \begin{vmatrix} 1 & 2 & 3 \\ 4-4 & 5-5 & 6-6 \\ 7 & 8 & 9 \end{vmatrix}$

$= \begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 7 & 8 & 9 \end{vmatrix} = 0$

$\because R_2$ Consists of all elments zero's.

Q.3 Show that

(i) $\begin{vmatrix} a_{11} & a_{12} & a_{13}+\alpha_{13} \\ a_{21} & a_{22} & a_{23}+\alpha_{23} \\ a_{31} & a_{32} & a_{33}+\alpha_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \alpha_{13} \\ a_{21} & a_{22} & \alpha_{23} \\ a_{31} & a_{32} & \alpha_{33} \end{vmatrix}$

$RHS = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \alpha_{13} \\ a_{21} & a_{22} & \alpha_{23} \\ a_{31} & a_{32} & \alpha_{33} \end{vmatrix}$

According to the Property (7)

$= \begin{vmatrix} a_{11} & a_{12} & a_{13}+\alpha_{13} \\ a_{21} & a_{22} & a_{23}+\alpha_{23} \\ a_{31} & a_{32} & a_{33}+\alpha_{33} \end{vmatrix}$ (Proved).

Alternate
(Expand both sides to get result.)

(ii) $\begin{vmatrix} 2 & 3 & 0 \\ 3 & 9 & 6 \\ 2 & 15 & 1 \end{vmatrix} = 9\begin{vmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \\ 2 & 5 & 1 \end{vmatrix}$

$LHS = \begin{vmatrix} 2 & 3 & 0 \\ 3 & 9 & 6 \\ 2 & 15 & 1 \end{vmatrix}$

Taking 3 Common from C_2

$= 3\begin{vmatrix} 2 & 1 & 0 \\ 3 & 3 & 6 \\ 2 & 5 & 1 \end{vmatrix}$

Taking 3 Common from R_2

$= 3 \cdot 3\begin{vmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \\ 2 & 5 & 1 \end{vmatrix}$

$= 9\begin{vmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \\ 2 & 5 & 1 \end{vmatrix}$ (Proved)

(iii) $\begin{vmatrix} a+l & a & a \\ a & a+l & a \\ a & a & a+l \end{vmatrix} = l^2(3a+l)$

$LHS = \begin{vmatrix} a+l & a & a \\ a & a+l & a \\ a & a & a+l \end{vmatrix}$

By $(C_2 + C_3) + C_1$

$\begin{vmatrix} 3a+l & a & a \\ 3a+l & a+l & a \\ 3a+l & a & a+l \end{vmatrix}$

Taking $(3a+l)$ Common from C_1

$(3a+l)\begin{vmatrix} 1 & a & a \\ 1 & a+l & a \\ 1 & a & a+l \end{vmatrix}$

By $(R_2 - R_1)$ and $(R_3 - R_1)$

$(3a+l)\begin{vmatrix} 1 & a & a \\ 0 & l & 0 \\ 0 & 0 & l \end{vmatrix}$

Expanding w.r.t Ist Column.

$(3a+l)\left\{1\begin{vmatrix} l & 0 \\ 0 & l \end{vmatrix} - 0 + 0\right\}$

$(3a+l)\{1(l^2-0) - 0 + 0\}$

$(3a+l)\,l^2$ (Proved)

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(iv) $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ yz & zx & xy \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$

$LHS = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ yz & zx & xy \end{vmatrix}$

Multiply and divide by xyz

$= \dfrac{xyz}{xyz}\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ yz & zx & xy \end{vmatrix}$

Multiply x by C_1, y by C_2, z by C_3

$= \dfrac{1}{xyz}\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ xyz & xyz & xyz \end{vmatrix}$

Taking xyz Common from R_3

$= \dfrac{\cancel{xyz}}{\cancel{xyz}}\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ 1 & 1 & 1 \end{vmatrix}$

$(-1)^2\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$

$= \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$ (Proved)

(v) $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc$

$LHS = \begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$

Expanding from R_1

$= (b+c)\begin{vmatrix} c+a & b \\ c & a+b \end{vmatrix} - a\begin{vmatrix} b & b \\ c & a+b \end{vmatrix} + a\begin{vmatrix} b & c+a \\ c & c \end{vmatrix}$

$= (b+c)\{ac + a^2 + \cancel{bc} + ab - \cancel{bc}\}$

$-a\{ab + b^2 - bc\} + a\{bc - c^2 - ca\}$

$= (b+c)(ac+ab+a^2) - a(ab+b^2-bc)$

$+ a(bc - c^2 - ca)$

$= abc + \cancel{ac^2} + \cancel{ab^2} + abc + \cancel{a^2b} + \cancel{a^2c}$

$- \cancel{a^2b} - \cancel{ab^2} + abc + abc - \cancel{ac^2} - \cancel{ca^2}$

$= 4abc$ (Proved)

Adjoint of 3x3 or higher matrix:-

If A is 3x3 or higher order matrix then

$\text{Adj}A = [\text{Cofactor matrix of } A]^t$.

(vi) $\begin{vmatrix} b & -1 & a \\ a & b & 0 \\ 1 & a & b \end{vmatrix} = a^3 + b^3$

$LHS = \begin{vmatrix} b & -1 & a \\ a & b & 0 \\ 1 & a & b \end{vmatrix}$

Expanding w.r.t 1st Row

$= b\begin{vmatrix} b & 0 \\ a & b \end{vmatrix} + 1\begin{vmatrix} a & 0 \\ 1 & b \end{vmatrix} + a\begin{vmatrix} a & b \\ 1 & a \end{vmatrix}$

$= b(b^2 - 0) + 1(ab - 0) + a(a^2 - b)$

$= b^3 - 0 + \cancel{ab} - 0 + a^3 - \cancel{ab}$

$= a^3 + b^3$ (Proved)

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(vii) $\begin{vmatrix} r\cos\phi & 1 & -\sin\phi \\ 0 & 1 & 0 \\ r\sin\phi & 0 & \cos\phi \end{vmatrix} = r$

LHS: Expanding w.r.t 1st row

$= r\cos\phi \begin{vmatrix} 1 & 0 \\ 0 & \cos\phi \end{vmatrix} - 1\begin{vmatrix} 0 & 0 \\ r\sin\phi & \cos\phi \end{vmatrix} - \sin\phi \begin{vmatrix} 0 & 1 \\ r\sin\phi & 0 \end{vmatrix}$

$= r\cos\phi(\cos\phi - 0) - 1(0) - \sin\phi(0 - r\sin\phi)$

$= r\cos^2\phi - 0 + r\sin^2\phi$

$= r(\sin^2\phi + \cos^2\phi)$

$= r(1) = r$ (proved)

(viii) $\begin{vmatrix} a & b+c & a+b \\ b & c+a & b+c \\ c & a+b & c+a \end{vmatrix} = a^3+b^3+c^3-3abc$

LHS $= \begin{vmatrix} a & b+c & a+b \\ b & c+a & b+c \\ c & a+b & c+a \end{vmatrix}$

By $C_1 + C_2$, we get

$\begin{vmatrix} a+b+c & b+c & a+b \\ b+c+a & c+a & b+c \\ c+a+b & a+b & c+a \end{vmatrix}$

Taking $(a+b+c)$ common from C_1

$(a+b+c)\begin{vmatrix} 1 & b+c & a+b \\ 1 & c+a & b+c \\ 1 & a+b & c+a \end{vmatrix}$

By $R_2 - R_1$ and $R_3 - R_1$

$(a+b+c)\begin{vmatrix} 1 & b+c & a+b \\ 0 & a-b & c-a \\ 0 & a-c & c-b \end{vmatrix}$

Expanding w.r.t C_1

$(a+b+c)\left\{1\begin{vmatrix} a-b & c-a \\ a-c & c-b \end{vmatrix} - 0 + 0\right\}$

$(a+b+c)\{(a-b)(c-b) - (c-a)(a-c)\}$

$= (a+b+c)\{ac - ab - bc + b^2 - (ca - a^2 - c^2 + ac)\}$

$= (a+b+c)\{ac - ab - bc + b^2 - ca + a^2 + c^2 - ac\}$

$(a+b+c)(a^2+b^2+c^2 - ab - bc - ca)$

$= a^3+b^3+c^3-3abc$ (proved)

(ix) $\begin{vmatrix} a+\lambda & b & c \\ a & b+\lambda & c \\ a & b & c+\lambda \end{vmatrix} = \lambda^2(a+b+c+\lambda)$

LHS = By $C_1 + (C_2 + C_3)$

$\begin{vmatrix} a+\lambda+b+c & b & c \\ a+b+\lambda+c & b+\lambda & c \\ a+b+\lambda+c & b & c+\lambda \end{vmatrix}$

Taking $(a+b+c+\lambda)$ from C_1

$(a+b+c+\lambda)\begin{vmatrix} 1 & b & c \\ 1 & b+\lambda & c \\ 1 & b & c+\lambda \end{vmatrix}$

By $R_2 - R_1$ and $R_3 - R_1$

$(a+b+c+\lambda)\begin{vmatrix} 1 & b & c \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{vmatrix}$

Expanding w.r.t C_1

$(a+b+c+\lambda)\left\{1\begin{vmatrix} \lambda & 0 \\ 0 & \lambda \end{vmatrix} - 0 + 0\right\}$

$(a+b+c+\lambda)\,1(\lambda^2 - 0)$

$= \lambda^2(a+b+c+\lambda)$ (Proved)

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TAHIR

(x) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

LHS = By $C_1 - C_2$, $C_2 - C_3$

$\begin{vmatrix} 1-1 & 1-1 & 1 \\ a-b & b-c & c \\ a^2-b^2 & b^2-c^2 & c^2 \end{vmatrix}$

$\begin{vmatrix} 0 & 0 & 1 \\ (a-b) & (b-c) & c \\ (a-b)(a+b) & (b-c)(b+c) & c^2 \end{vmatrix}$

Taking $(a-b)$ from C_1 and $(b-c)$ from C_2 Common

$(a-b)(b-c)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ a+b & b+c & c^2 \end{vmatrix}$

Expanding w.r.t Ist row

$(a-b)(b-c)\left\{0 - 0 + 1\begin{vmatrix} 1 & 1 \\ a+b & b+c \end{vmatrix}\right\}$

$(a-b)(b-c)\,1(\cancel{b}+c-a-\cancel{b})$

$=(a-b)(b-c)(c-a)$ (Proved)

(xi) $\begin{vmatrix} b+c & a & a^2 \\ c+a & b & b^2 \\ a+b & c & c^2 \end{vmatrix} = (a+b+c)(a-b)(b-c)(c-a)$

LHS = By $C_1 + C_2$

$\begin{vmatrix} a+b+c & a & a^2 \\ a+b+c & b & b^2 \\ a+b+c & c & c^2 \end{vmatrix}$

Taking $(a+b+c)$ Common from C_1

$(a+b+c)\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

By $R_1 - R_2$, $R_2 - R_3$

$(a+b+c)\begin{vmatrix} 0 & a-b & a^2-b^2 \\ 0 & b-c & b^2-c^2 \\ 1 & c & c^2 \end{vmatrix}$

Taking $(a-b)$ from R_1 and $(b-c)$ from R_2

$(a+b+c)(a-b)(b-c)\begin{vmatrix} 0 & 1 & a+b \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix}$

Expanding w.r.t C_1

$(a+b+c)(a-b)(b-c)\left\{0 - 0 + 1\begin{vmatrix} 1 & a+b \\ 1 & b+c \end{vmatrix}\right\}$

$(a+b+c)(a-b)(b-c)\,1(\cancel{b}+c-a-\cancel{b})$

$(a+b+c)(a-b)(b-c)(c-a)$ (Proved)

Q.4 $A=\begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2 & -2 & 1 \end{bmatrix}$ $B=\begin{bmatrix} 5 & -2 & 5 \\ 3 & -1 & 4 \\ -2 & 1 & -2 \end{bmatrix}$

(i) A_{12}, A_{22}, A_{32} and $|A| = ?$

$\because A_{ij} = (-1)^{i+j} M_{ij}$

$A_{12} = (-1)^{1+2}\begin{vmatrix} 0 & 0 \\ -2 & 1 \end{vmatrix}$

$= (-1)^3(0-0)$

$= -1(0) = 0$ ✓

$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & -3 \\ -2 & 1 \end{vmatrix}$

$= (-1)^4(1-6)$

$= 1(-5) = -5$ ✓

$A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & -3 \\ 0 & 0 \end{vmatrix}$

$= (-1)^5(0+0) = 0$

$|A| = 1\begin{vmatrix} -2 & 0 \\ -2 & 1 \end{vmatrix} - 0\begin{vmatrix} 2 & -3 \\ -2 & 1 \end{vmatrix} - 2\begin{vmatrix} 2 & -3 \\ -2 & 0 \end{vmatrix}$

$= 1(-2+0) - 0 - 2(0-6)$

$= -2 + 12 = 10$ Ans.

Alternate Using Cofactors

$|A| = a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32}$

$|A| = 2(0) + (-2)(-5) + (-2)(0)$

$= 0 + 10 + 0$

$= 10$ Ans

(ii) B_{21}, B_{22}, B_{23} and $|B| = ?$

$B_{21} = (-1)^{2+1}\begin{vmatrix} -2 & 5 \\ 1 & -2 \end{vmatrix}$

$= (-1)^3(4-5)$

$= (-1)(-1) = 1$

$B_{22} = (-1)^{2+2}\begin{vmatrix} 5 & 5 \\ -2 & -2 \end{vmatrix}$

$= (-1)^4(-10+10)$

$= (-1)^4(0) = 0$

$B_{23} = (-1)^{2+3}\begin{vmatrix} 5 & -2 \\ -2 & 1 \end{vmatrix}$

$= (-1)^5(5-4)$

$= (-1)(+1) = -1$

$|B| = b_{21}B_{21} + b_{22}B_{22} + b_{23}B_{23}$

$|B| = 3(1) + (-1)(0) + (4)(-1)$

$= 3 - 0 - 4 = -1$ Ans

Alternate:

$|B| = 5\begin{vmatrix} -1 & 4 \\ 1 & -2 \end{vmatrix} + 2\begin{vmatrix} 3 & 4 \\ -2 & -2 \end{vmatrix} + 5\begin{vmatrix} 3 & -1 \\ -2 & 1 \end{vmatrix}$

$= 5(2-4) + 2(-6+8) + 5(3-2)$

$= 5(-2) + 2(2) + 5(1)$

$= -10 + 4 + 5 = -10 + 9$

$= -1$ Ans

Q.5 Without Expansion Verify that:

(i) $\begin{vmatrix} \alpha & \beta+\gamma & 1 \\ \beta & \alpha+\gamma & 1 \\ \gamma & \alpha+\beta & 1 \end{vmatrix} = 0$

LHS= By $C_1 + C_2$ we get

$\begin{vmatrix} \alpha+\beta+\gamma & \beta+\gamma & 1 \\ \alpha+\beta+\gamma & \gamma+\alpha & 1 \\ \alpha+\beta+\gamma & \alpha+\beta & 1 \end{vmatrix}$

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Taking $(\alpha+\beta+\gamma)$ Common from C_1

$(\alpha+\beta+\gamma)\begin{vmatrix} 1 & \beta+\gamma & 1 \\ 1 & \gamma+\alpha & 1 \\ 1 & \alpha+\beta & 1 \end{vmatrix} = 0$

$\because$ Two Colums are identical so $|A| = 0$

(ii) $\begin{vmatrix} 1 & 2 & 3x \\ 2 & 3 & 6x \\ 3 & 5 & 9x \end{vmatrix} = 0$

LHS= Taking $3x$ Common from C_3

$3x\begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 3 & 5 & 3 \end{vmatrix} = 0$

$\because$ Two Columns are identical so $|A| = 0$

(iii) $\begin{vmatrix} 1 & a^2 & a/bc \\ 1 & b^2 & b/ca \\ 1 & c^2 & c/ab \end{vmatrix} = 0$

LHS= Multiply and dividing by abc

$= \frac{abc}{abc}\begin{vmatrix} 1 & a^2 & a/bc \\ 1 & b^2 & b/ac \\ 1 & c^2 & c/ab \end{vmatrix}$

= Multiplying C_3 by abc

$\frac{1}{abc}\begin{vmatrix} 1 & a^2 & \frac{a}{bc} \times abc \\ 1 & b^2 & b/ac \times abc \\ 1 & c^2 & c/ab \times abc \end{vmatrix}$

$\frac{1}{abc}\begin{vmatrix} 1 & a^2 & a^2 \\ 1 & b^2 & b^2 \\ 1 & c^2 & c^2 \end{vmatrix} = 0$

$\because$ Two Columns are identical so $|A| = 0$

(iv) $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$

LHS = By $C_1 + (C_2 + C_3)$

$\begin{vmatrix} a-b+b-c+c-a & b-c & c-a \\ b-c+c-a+a-b & c-a & a-b \\ c-a+a-b+b-c & a-b & b-c \end{vmatrix}$

$\begin{vmatrix} 0 & b-c & c-a \\ 0 & c-a & a-b \\ 0 & a-b & b-c \end{vmatrix} = 0$

$\because$ C_1 is zero so $|A| = 0$

(v) $\begin{vmatrix} bc & ca & ab \\ 1/a & 1/b & 1/c \\ a & b & c \end{vmatrix} = 0$

LHS = Multiplying and dividing by abc

$= \frac{abc}{abc} \begin{vmatrix} bc & ca & ab \\ 1/a & 1/b & 1/c \\ a & b & c \end{vmatrix}$

By $R_2 \times (abc)$, we get

$= \frac{1}{abc} \begin{vmatrix} bc & ca & ab \\ \frac{1}{a}\times abc & \frac{1}{b}\times abc & \frac{1}{c}\times abc \\ a & b & c \end{vmatrix}$

$= \frac{1}{abc} \begin{vmatrix} bc & ca & ab \\ bc & ca & ab \\ a & b & c \end{vmatrix} = 0$

$\because R_1 \equiv R_2$ so $|A| = 0$

(vi) $\begin{vmatrix} mn & l & l^2 \\ nl & m & m^2 \\ lm & n & n^2 \end{vmatrix} = \begin{vmatrix} 1 & l^2 & l^3 \\ 1 & m^2 & m^3 \\ 1 & n^2 & n^3 \end{vmatrix}$

LHS $= \begin{vmatrix} mn & l & l^2 \\ nl & m & m^2 \\ lm & n & n^2 \end{vmatrix}$

$= \frac{lmn}{lmn} \begin{vmatrix} mn & l & l^2 \\ nl & m & m^2 \\ lm & n & n^2 \end{vmatrix}$

By $R_1 \times l$, $R_2 \times m$, $R_3 \times n$

$= \frac{1}{lmn} \begin{vmatrix} lmn & l^2 & l^3 \\ lmn & m^2 & m^3 \\ lmn & n^2 & n^3 \end{vmatrix}$

Taking lmn Common from C_1

$= \frac{\cancel{lmn}}{\cancel{lmn}} \begin{vmatrix} 1 & l^2 & l^3 \\ 1 & m^2 & m^3 \\ 1 & n^2 & n^3 \end{vmatrix}$

$= \begin{vmatrix} 1 & l^2 & l^3 \\ 1 & m^2 & m^3 \\ 1 & n^2 & n^3 \end{vmatrix}$ (Proved)

(vii) $\begin{vmatrix} 2a & 2b & 2c \\ a+b & 2b & b+c \\ a+c & b+c & 2c \end{vmatrix} = 0$

LHS $= \begin{vmatrix} 2a & 2b & 2c \\ a+b & 2b & b+c \\ a+c & b+c & 2c \end{vmatrix}$

$= 2 \begin{vmatrix} a & b & c \\ a+b & 2b & b+c \\ a+c & b+c & 2c \end{vmatrix}$ 2 Common from R_1

By $C_2 - C_1$ and $C_3 - C_1$

$= 2 \begin{vmatrix} a & b-a & c-a \\ a+b & 2b-a-b & b+c-a-b \\ a+c & b+c-a-c & 2c-a-c \end{vmatrix}$

$= 2 \begin{vmatrix} a & b-a & c-a \\ a+b & b-a & c-a \\ a+c & b-a & c-a \end{vmatrix}$

$= 2(b-a)(c-a) \begin{vmatrix} a & 1 & 1 \\ a+b & 1 & 1 \\ a+c & 1 & 1 \end{vmatrix} = 0$

$\because C_2 \equiv C_3$ so $|A| = 0$

$\therefore \begin{vmatrix} 2a & 2b & 2c \\ a+b & 2b & b+c \\ a+c & b+c & 2c \end{vmatrix} = 0$

 (Proved)

(viii) $\begin{vmatrix} 7 & 2 & 6 \\ 6 & 3 & 2 \\ -3 & 5 & 1 \end{vmatrix} = \begin{vmatrix} 7 & 2 & 7 \\ 6 & 3 & 5 \\ -3 & 5 & -3 \end{vmatrix} + \begin{vmatrix} 7 & 2 & -1 \\ 6 & 3 & -3 \\ -3 & 5 & 4 \end{vmatrix}$

RHS: $\begin{vmatrix} 7 & 2 & 7 \\ 6 & 3 & 5 \\ -3 & 5 & -3 \end{vmatrix} + \begin{vmatrix} 7 & 2 & -1 \\ 6 & 3 & -3 \\ -3 & 5 & 4 \end{vmatrix}$

Using Property (7) where C_1 and C_2 are same

$= \begin{vmatrix} 7 & 2 & 7+(-1) \\ 6 & 3 & 5+(-3) \\ -3 & 5 & -3+4 \end{vmatrix}$

$= \begin{vmatrix} 7 & 2 & 6 \\ 6 & 3 & 2 \\ -3 & 5 & 1 \end{vmatrix}$ (Proved)

(ix) $\begin{vmatrix} -a & 0 & c \\ 0 & a & -b \\ b & -c & 0 \end{vmatrix} = 0$

LHS $= \begin{vmatrix} -a & 0 & c \\ 0 & a & -b \\ b & -c & 0 \end{vmatrix}$

$= \dfrac{abc}{abc} \begin{vmatrix} -a & 0 & c \\ 0 & a & -b \\ b & -c & 0 \end{vmatrix}$

By $C_1 \times c,\ C_2 \times b,\ C_3 \times a$

$= \dfrac{1}{abc} \begin{vmatrix} -ac & 0 & ca \\ 0 & ab & -ab \\ bc & -bc & 0 \end{vmatrix}$

By $C_1 + (C_2 + C_3)$

$= \dfrac{1}{abc} \begin{vmatrix} -ac+0+ca & 0 & ca \\ 0+ab-ab & ab & -ab \\ bc-bc+0 & -bc & 0 \end{vmatrix}$

$= \dfrac{1}{abc} \begin{vmatrix} 0 & 0 & ca \\ 0 & ab & -ab \\ 0 & -bc & 0 \end{vmatrix}$

$\because C_1$ contains all entries zero

so determinant is zero.

Hence $\begin{vmatrix} -a & 0 & c \\ 0 & a & -b \\ b & -c & 0 \end{vmatrix} = 0$

Q.6 Find the value of x if

(i) $\begin{vmatrix} 3 & 1 & x \\ -1 & 3 & 4 \\ x & 1 & 0 \end{vmatrix} = -30$

$\Rightarrow 3\begin{vmatrix} 3 & 4 \\ 1 & 0 \end{vmatrix} - 1\begin{vmatrix} -1 & 4 \\ x & 0 \end{vmatrix} + x\begin{vmatrix} -1 & 3 \\ x & 1 \end{vmatrix} = -30$

$\Rightarrow 3(0-4) - 1(0-4x) + x(-1-3x) = -30$

$\Rightarrow -12 + 4x - x - 3x^2 = -30$

$\Rightarrow -3x^2 + 3x - 12 = -30$

$\Rightarrow 3x^2 - 3x + 12 - 30 = 0$

$\Rightarrow 3x^2 - 3x - 18 = 0$

$\Rightarrow x^2 - x - 6 = 0$ Dividing by 3

$\Rightarrow (x-3)(x+2) = 0$

$\Rightarrow x - 3 = 0$ or $x + 2 = 0$

$\Rightarrow x = 3$ or $x = -2$

(ii) $\begin{vmatrix} 1 & x-1 & 3 \\ -1 & x+1 & 2 \\ 2 & -2 & x \end{vmatrix} = 0$

$\Rightarrow 1\begin{vmatrix} x+1 & 2 \\ -2 & x \end{vmatrix} - (x-1)\begin{vmatrix} -1 & 2 \\ 2 & x \end{vmatrix} + 3\begin{vmatrix} -1 & x+1 \\ 2 & -2 \end{vmatrix} = 0$

$\Rightarrow 1(x^2+x+4) - (x-1)(-x-4) + 3(\cancel{2} - 2x - \cancel{2}) = 0$

$\Rightarrow x^2 \cancel{+x} \cancel{+4} + x^2 + 4x \cancel{-x} \cancel{-4} - 6x = 0$

$\Rightarrow 2x^2 - 2x = 0$

$\Rightarrow 2x(x-1) = 0$

$\Rightarrow 2x = 0$ or $x - 1 = 0$

$\Rightarrow x = 0$ or $x = 1$

So $x = 0$ or $x = 1$

(iii) $\begin{vmatrix} 1 & 2 & 1 \\ 2 & x & 2 \\ 3 & 6 & x \end{vmatrix} = 0$

$\Rightarrow 1\begin{vmatrix} x & 2 \\ 6 & x \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 3 & x \end{vmatrix} + 1\begin{vmatrix} 2 & x \\ 3 & 6 \end{vmatrix} = 0$

$\Rightarrow 1(x^2-12) - 2(2x-6) + 1(12-3x) = 0$

$\Rightarrow x^2 \cancel{-12} - 4x \cancel{+12} + 12 - 3x = 0$

$\Rightarrow x^2 - 7x + 12 = 0$

$\Rightarrow (x-3)(x-4) = 0$

$\Rightarrow x-3=0 \quad \text{or} \quad x-4=0$

$\Rightarrow x=3 \quad \text{or} \quad x=4$

Q.7 (i) Evaluate the followings:

$$\begin{vmatrix} 3 & 4 & 2 & 7 \\ 2 & 5 & 0 & 3 \\ 1 & 2 & -3 & 5 \\ 4 & 1 & -2 & 6 \end{vmatrix}$$

Expanding from R_1

$$= 3\begin{vmatrix} 5 & 0 & 3 \\ 2 & -3 & 5 \\ 1 & -2 & 6 \end{vmatrix} - 4\begin{vmatrix} 2 & 0 & 3 \\ 1 & -3 & 5 \\ 4 & -2 & 6 \end{vmatrix} + 2\begin{vmatrix} 2 & 5 & 3 \\ 1 & 2 & 5 \\ 4 & 1 & 6 \end{vmatrix} - 7\begin{vmatrix} 2 & 5 & 0 \\ 1 & 2 & -3 \\ 4 & 1 & -2 \end{vmatrix}$$

$$= 3\left\{5\begin{vmatrix} -3 & 5 \\ -2 & 6 \end{vmatrix} - 0 + 3\begin{vmatrix} 2 & -3 \\ 1 & -2 \end{vmatrix}\right\} - 4\left\{2\begin{vmatrix} -3 & 5 \\ -2 & 6 \end{vmatrix} - 0 + 3\begin{vmatrix} 1 & -3 \\ 4 & -2 \end{vmatrix}\right\} + 2\left\{2\begin{vmatrix} 2 & 5 \\ 1 & 6 \end{vmatrix} - 5\begin{vmatrix} 1 & 5 \\ 4 & 6 \end{vmatrix} + 3\begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix}\right\}$$
$$-7\left\{2\begin{vmatrix} 2 & -3 \\ 1 & -2 \end{vmatrix} - 5\begin{vmatrix} 1 & -3 \\ 4 & -2 \end{vmatrix} + 0\right\}$$

$$= 3\{5(-18+10) - 0 + 3(-4+3)\} - 4\{2(-18+10) - 0 + 3(-2+12)\} + 2\{2(12-5) - 5(6-20) + 3(1-8)\}$$
$$-7\{2(-4+3) - 5(-2+12) + 0\}$$

$$= 3\{5(-8) - 0 + 3(-1)\} - 4\{2(-8) - 0 + 3(10)\} + 2\{2(7) - 5(-14) + 3(-7)\} - 7\{2(-1) - 5(10) + 0\}$$

$$= 3[-40-3] - 4[-16+30] + 2[14+70-21] - 7[-2-50+0]$$

$$= 3[-43] - 4(14) + 2[63] - 7[-52]$$

$$= -129 - 56 + 126 + 364$$

$$= 305 \quad \text{Ans.}$$

(28)

Alternatively

$$\begin{vmatrix} 3 & 4 & 2 & 7 \\ 2 & 5 & 0 & 3 \\ 1 & 2 & -3 & 5 \\ 4 & 1 & -2 & 6 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -2 & 11 & -8 \\ 0 & 1 & 6 & -7 \\ 1 & 2 & -3 & 5 \\ 0 & -7 & 10 & -14 \end{vmatrix}$$ By $R_1 - 3R_3$, $R_2 - 2R_3$, $R_4 - 4R_3$

Expanding from C_1

$$= 0 - 0 + 1\begin{vmatrix} -2 & 11 & -8 \\ 1 & 6 & -7 \\ -7 & 10 & -14 \end{vmatrix} - 0$$

$$= 0 - 0 + \left\{-2\begin{vmatrix} 6 & -7 \\ 10 & -14 \end{vmatrix} - 11\begin{vmatrix} 1 & -7 \\ -7 & -14 \end{vmatrix} - 8\begin{vmatrix} 1 & 6 \\ -7 & 10 \end{vmatrix}\right\}$$

$$= 0 - 0 - 2(-84+70) - 11(-14-49) - 8(10+42)$$

$$= -2(-14) - 11(-63) - 8(52) - 0$$

$$= 28 + 693 - 416$$

$$= 721 - 416 = 305$$

$$= 305 \quad \text{Ans.}$$

(ii) $\begin{vmatrix} 2 & 3 & 1 & -1 \\ 4 & 0 & 2 & 1 \\ 5 & 2 & -1 & 6 \\ 3 & -7 & 2 & -2 \end{vmatrix}$

By Interchanging C_2 and C_3

$$(-1)\begin{vmatrix} 2 & 1 & 3 & -1 \\ 4 & 2 & 0 & 1 \\ 5 & -1 & 2 & 6 \\ 3 & 2 & -7 & -2 \end{vmatrix}$$

$$(-1)(-1)\begin{vmatrix} 1 & 2 & 3 & -1 \\ 2 & 4 & 0 & 1 \\ -1 & 5 & 2 & 6 \\ 2 & 3 & -7 & -2 \end{vmatrix}$$ By interchanging C_1 and C_2

$$\begin{vmatrix} 1 & 2 & 3 & -1 \\ 2 & 4 & 0 & 1 \\ -1 & 5 & 2 & 6 \\ 2 & 3 & -7 & -2 \end{vmatrix}$$

By R_2-2R_1, R_3+R_1, R_4-2R_1

$$\begin{vmatrix} 1 & 2 & 3 & -1 \\ 0 & 0 & -6 & 3 \\ 0 & 7 & 5 & 5 \\ 0 & -1 & -13 & 0 \end{vmatrix}$$

Expanding w.r.t C_1

$$1\begin{vmatrix} 0 & -6 & 3 \\ 7 & 5 & 5 \\ -1 & -13 & 0 \end{vmatrix} - 0 + 0 - 0$$

$$\begin{vmatrix} 0 & -6 & 3 \\ 7 & 5 & 5 \\ -1 & -13 & 0 \end{vmatrix}$$

Expanding from R_1

$$0 + 6\begin{vmatrix} 7 & 5 \\ -1 & 0 \end{vmatrix} + 3\begin{vmatrix} 7 & 5 \\ -1 & -13 \end{vmatrix}$$

$$0 + 6(0+5) + 3(-91+5)$$

$$30 + (-258) = -228$$

Ans.

(iii) $\begin{vmatrix} -3 & 9 & 1 & 1 \\ 0 & 3 & -1 & 2 \\ 9 & 7 & -1 & 1 \\ -2 & 0 & 1 & -1 \end{vmatrix}$

By interchanging C_2 and C_3

$$(-1)\begin{vmatrix} -3 & 1 & 9 & 1 \\ 0 & -1 & 3 & 2 \\ 9 & -1 & 7 & 1 \\ -2 & 1 & 0 & -1 \end{vmatrix}$$

Again interchanging C_1 and C_2

$$(-1)^2\begin{vmatrix} 1 & -3 & 9 & 1 \\ -1 & 0 & 3 & 2 \\ -1 & 9 & 7 & 1 \\ 1 & -2 & 0 & -1 \end{vmatrix}$$

$$\begin{vmatrix} 1 & -3 & 9 & 1 \\ -1 & 0 & 3 & 2 \\ -1 & 9 & 7 & 1 \\ 1 & -2 & 0 & -1 \end{vmatrix}$$

By R_2+R_1, R_3+R_1, R_4-R_1

$$\begin{vmatrix} 1 & -3 & 9 & 1 \\ 0 & -3 & 12 & 3 \\ 0 & 6 & 16 & 2 \\ 0 & 1 & -9 & -2 \end{vmatrix}$$

Expanding from C_1

$$1\begin{vmatrix} -3 & 12 & 3 \\ 6 & 16 & 2 \\ 1 & -9 & -2 \end{vmatrix} - 0 + 0 - 0$$

Expanding from C_1

$$-3\begin{vmatrix} 16 & 2 \\ -9 & -2 \end{vmatrix} - 6\begin{vmatrix} 12 & 3 \\ -9 & -2 \end{vmatrix} + 1\begin{vmatrix} 12 & 3 \\ 16 & 2 \end{vmatrix}$$

$$-3(-32+18) - 6(-24+27) + 1(24-48)$$

$$-3(-14) - 6(3) + 1(-24)$$

$$42 - 18 - 24 \Rightarrow 42 - 42$$

$$42 - 42 = 0$$

Ans.

(30)

Q.8
$$\begin{vmatrix} x & 1 & 1 & 1 \\ 1 & x & 1 & 1 \\ 1 & 1 & x & 1 \\ 1 & 1 & 1 & x \end{vmatrix} = (x+3)(x-1)^3$$

LHS:
$$\begin{vmatrix} x & 1 & 1 & 1 \\ 1 & x & 1 & 1 \\ 1 & 1 & x & 1 \\ 1 & 1 & 1 & x \end{vmatrix}$$

By $C_1 + (C_2 + C_3 + C_4)$, we get

$$\begin{vmatrix} (x+3) & 1 & 1 & 1 \\ (x+3) & x & 1 & 1 \\ (x+3) & 1 & x & 1 \\ (x+3) & 1 & 1 & x \end{vmatrix}$$

$(x+3)$ taking common from C_1 and

$$(x+3)\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & x & 1 & 1 \\ 1 & 1 & x & 1 \\ 1 & 1 & 1 & x \end{vmatrix}$$

Now By $R_2 - R_1$, $R_3 - R_1$, $R_4 - R_1$

$$(x+3)\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & x-1 & 0 & 0 \\ 0 & 0 & x-1 & 0 \\ 0 & 0 & 0 & (x-1) \end{vmatrix}$$

$$(x+3)\begin{vmatrix} x-1 & 0 & 0 \\ 0 & x-1 & 0 \\ 0 & 0 & x-1 \end{vmatrix} - 0 + 0 - 0$$ Expanding from C_1

$$(x+3)\left\{(x-1)\begin{vmatrix} x-1 & 0 \\ 0 & x-1 \end{vmatrix}\right\}$$ Expanding from C_1

$$(x+3)\left[(x-1)\left((x-1)^2 - 0\right)\right]$$

$$(x+3)(x-1)^3 \quad \text{(Proved)}$$

Q.9 Find $|AA^t|$ and $|A^tA|$ if.

(i) $A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 1 & 3 \end{bmatrix}$

$$A^t = \begin{bmatrix} 3 & 2 \\ 2 & 1 \\ -1 & 3 \end{bmatrix}$$

$$AA^t = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 2 & 1 \\ -1 & 3 \end{bmatrix}$$

$$AA^t = \begin{bmatrix} 9+4+1 & 6+2-3 \\ 6+2-3 & 4+1+9 \end{bmatrix}$$

$$AA^t = \begin{bmatrix} 14 & 5 \\ 5 & 14 \end{bmatrix}$$

$$|AA^t| = (14 \times 14) - (5 \times 5)$$
$$= 196 - 25 = 171$$

So $|AA^t| = 171$ — Ans

Now
$$A^tA = \begin{bmatrix} 3 & 2 \\ 2 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 2 & -1 \\ 2 & 1 & 3 \end{bmatrix}$$

$$A^tA = \begin{bmatrix} 9+4 & 6+2 & -3+6 \\ 6+2 & 4+1 & -2+3 \\ -3+6 & -2+3 & 1+9 \end{bmatrix}$$

$$A^tA = \begin{bmatrix} 13 & 8 & 3 \\ 8 & 5 & 1 \\ 3 & 1 & 10 \end{bmatrix}$$

$$|A^tA| = \begin{vmatrix} 13 & 8 & 3 \\ 8 & 5 & 1 \\ 3 & 1 & 10 \end{vmatrix}$$

Expanding from C_1

$$|A^tA| = 13\begin{vmatrix} 5 & 1 \\ 1 & 10 \end{vmatrix} - 8\begin{vmatrix} 8 & 3 \\ 1 & 10 \end{vmatrix} + 3\begin{vmatrix} 8 & 3 \\ 5 & 1 \end{vmatrix}$$

$$|A^tA| = 13(50-1) - 8(80-3) + 3(8-15)$$

$$|A^tA| = 13(49) - 8(77) + 3(-7)$$

$$= 637 - 616 - 21$$

$$= 637 - 637 = 0$$

$$|A^tA| = 0 \quad \text{Ans}$$

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(iii) $\begin{bmatrix} 3 & 4 \\ 2 & 1 \\ 1 & 1 \\ 2 & 3 \end{bmatrix}$

$A^t = \begin{pmatrix} 3 & 2 & 1 & 2 \\ 4 & 1 & 1 & 3 \end{pmatrix}$

$AA^t = \begin{pmatrix} 3 & 4 \\ 2 & 1 \\ 1 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 3 & 2 & 1 & 2 \\ 4 & 1 & 1 & 3 \end{pmatrix}$

$AA^t = \begin{pmatrix} 9+16 & 6+4 & 3+4 & 6+12 \\ 6+4 & 4+1 & 2+1 & 4+3 \\ 3+4 & 2+1 & 1+1 & 2+3 \\ 6+12 & 4+3 & 2+3 & 4+9 \end{pmatrix}$

$AA^t = \begin{pmatrix} 25 & 10 & 7 & 18 \\ 10 & 5 & 3 & 7 \\ 7 & 3 & 2 & 5 \\ 18 & 7 & 5 & 13 \end{pmatrix}$

$|AA^t| = \begin{vmatrix} 25 & 10 & 7 & 18 \\ 10 & 5 & 3 & 7 \\ 7 & 3 & 2 & 5 \\ 18 & 7 & 5 & 13 \end{vmatrix}$

By $C_4 - (C_3 + C_2)$

$|AA^t| = \begin{vmatrix} 25 & 10 & 7 & 1 \\ 10 & 5 & 3 & -1 \\ 7 & 3 & 2 & 0 \\ 18 & 7 & 5 & 1 \end{vmatrix}$

By $R_2 + R_1$ and $R_4 - R_1$

$|AA^t| = \begin{vmatrix} 25 & 10 & 7 & 1 \\ 35 & 15 & 10 & 0 \\ 7 & 3 & 2 & 0 \\ -7 & -3 & -2 & 0 \end{vmatrix}$

Taking (-1) Common from R_4, we get

$|AA^t| = (-1)\begin{vmatrix} 25 & 10 & 7 & 1 \\ 35 & 15 & 10 & 0 \\ 7 & 3 & 2 & 0 \\ 7 & 3 & 2 & 0 \end{vmatrix}$

$R_3 \equiv R_4$ so $|AA^t| = 0$

$\therefore |AA^t| = 0$

Now $A^tA = \begin{pmatrix} 3 & 2 & 1 & 2 \\ 4 & 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 2 & 1 \\ 1 & 1 \\ 2 & 3 \end{pmatrix}$

$A^tA = \begin{pmatrix} 9+4+1+4 & 12+2+1+6 \\ 12+2+1+6 & 16+1+1+9 \end{pmatrix}$

$A^tA = \begin{pmatrix} 18 & 21 \\ 21 & 27 \end{pmatrix}$

$|A^tA| = \begin{vmatrix} 18 & 21 \\ 21 & 27 \end{vmatrix}$

$|A^tA| = 486 - 441 = 45$ Ans.

Q.10 If A is a Square matrix of order 3, then Show the $|KA| = K^3|A|$

Proof: Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$\therefore KA = \begin{bmatrix} Ka_{11} & Ka_{12} & Ka_{13} \\ Ka_{21} & Ka_{22} & Ka_{23} \\ Ka_{31} & Ka_{32} & Ka_{33} \end{bmatrix}$

Taking determinant on both sides

$|KA| = \begin{vmatrix} Ka_{11} & Ka_{12} & Ka_{13} \\ Ka_{21} & Ka_{22} & Ka_{32} \\ Ka_{31} & Ka_{32} & Ka_{33} \end{vmatrix}$

$|KA| = K.K.K \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

By taking K Common from each row.

$|KA| = K^3 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$|KA| = K^3|A|$ (Proved)

Alternate:

Hint: Expand on both sides of the Question and you will get the require result.

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Q.11 Find λ if A, B are Singular:

$$A = \begin{bmatrix} 4 & \lambda & 3 \\ 7 & 3 & 6 \\ 2 & 3 & 1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 4 & \lambda & 3 \\ 7 & 3 & 6 \\ 2 & 3 & 1 \end{vmatrix}$$

Being Singular $|A| = 0$

$$\therefore \begin{vmatrix} 4 & \lambda & 3 \\ 7 & 3 & 6 \\ 2 & 3 & 1 \end{vmatrix} = 0$$

Expanding w.r.t C_1, we get

$$4\begin{vmatrix} 3 & 6 \\ 3 & 1 \end{vmatrix} - 7\begin{vmatrix} \lambda & 3 \\ 3 & 1 \end{vmatrix} + 2\begin{vmatrix} \lambda & 3 \\ 3 & 6 \end{vmatrix} = 0$$

$$4(3-18) - 7(\lambda - 9) + 2(6\lambda - 9) = 0$$

$$4(-15) - 7\lambda + 63 + 12\lambda - 18 = 0$$

$$-60 - 7\lambda + 63 + 12\lambda - 18 = 0$$

$$5\lambda - 15 = 0 \implies 5\lambda = 15$$

or $\lambda = 3$ Ans.

$$B = \begin{bmatrix} 5 & 1 & 2 & 0 \\ 8 & 2 & 5 & 1 \\ 3 & 2 & 0 & 1 \\ 2 & \lambda & -1 & 3 \end{bmatrix}$$

$$|B| = \begin{vmatrix} 5 & 1 & 2 & 0 \\ 8 & 2 & 5 & 1 \\ 3 & 2 & 0 & 1 \\ 2 & \lambda & -1 & 3 \end{vmatrix}$$

Being Singular matrix, $|B| = 0$

$$\therefore \begin{vmatrix} 5 & 1 & 2 & 0 \\ 8 & 2 & 5 & 1 \\ 3 & 2 & 0 & 1 \\ 2 & \lambda & -1 & 3 \end{vmatrix} = 0$$

Interchanging C_1 and C_2, we get

$$(-1)\begin{vmatrix} 1 & 5 & 2 & 0 \\ 2 & 8 & 5 & 1 \\ 2 & 3 & 0 & 1 \\ \lambda & 2 & -1 & 3 \end{vmatrix} = 0$$

By $C_2 - 5C_1$, $C_3 - 2C_1$

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & -2 & 1 & 1 \\ 2 & -7 & -4 & 1 \\ \lambda & 2-5\lambda & -1-2\lambda & 3 \end{vmatrix} = 0$$

Expanding from R_1, we get

$$\begin{vmatrix} -2 & 1 & 1 \\ -7 & -4 & 1 \\ 2-5\lambda & -1-2\lambda & 3 \end{vmatrix} - 0 + 0 - 0 = 0$$

Now again Expanding from R_1

$$-2\begin{vmatrix} -4 & 1 \\ -1-2\lambda & 3 \end{vmatrix} - 1\begin{vmatrix} -7 & 1 \\ 2-5\lambda & 3 \end{vmatrix} + 1\begin{vmatrix} -7 & -4 \\ 2-5\lambda & -1-2\lambda \end{vmatrix} = 0$$

$$-2(-12+1+2\lambda) - 1(-21-2+5\lambda) + 1(7+14\lambda+8-20\lambda) = 0$$

$$-2(-11+2\lambda) - 1(-23+5\lambda) + 1(15-6\lambda) = 0$$

$$22 - 4\lambda + 23 - 5\lambda + 15 - 6\lambda = 0$$

$$60 - 15\lambda = 0 \implies 15\lambda = 60$$

$$\lambda = \frac{60}{15} = 4$$

$\boxed{\lambda = 4}$ Ans.

Q.12 Which of the matrices is Singular

$$\text{(ii)} \begin{vmatrix} 1 & 0 & 3 \\ 3 & 1 & -1 \\ 0 & 2 & 4 \end{vmatrix}$$

Expanding from R_1, we get

$$1\begin{vmatrix} 1 & -1 \\ 2 & 4 \end{vmatrix} - 0 + 3\begin{vmatrix} 3 & 1 \\ 0 & 2 \end{vmatrix}$$

$$1(4+2) - 0 + 3(6-0)$$

$$1(6) - 0 + 3(6) \implies 6 - 0 + 18$$

$= 24 \neq 0$ So

The matrix is non Singular.

(ii) $\begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{vmatrix}$

Expanding from R_1

$2\begin{vmatrix} 1 & 0 \\ -3 & 5 \end{vmatrix} - 3\begin{vmatrix} 1 & 0 \\ 2 & 5 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix}$

$2(5-0) - 3(5-0) - 1(-3-2)$

$= 10 - 15 + 5 \Rightarrow 15 - 15 = 0$

$\therefore$ determinant $= 0$ so matrix is Singular.

(iii) $\begin{vmatrix} 1 & 1 & 2 & -1 \\ 1 & 2 & -1 & -3 \\ 2 & 3 & 1 & 2 \\ 3 & -1 & 3 & 4 \end{vmatrix}$

By $C_2 - C_1$, $C_3 - 2C_1$, $C_4 + C_1$

$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & -3 & -2 \\ 2 & 1 & -3 & 4 \\ 3 & -4 & -3 & 7 \end{vmatrix}$

Expanding from R_1

$1\begin{vmatrix} 1 & -3 & -2 \\ 1 & -3 & 4 \\ -4 & -3 & 7 \end{vmatrix} - 0 + 0 - 0$

Taking -3 Common from C_2

$(-3)\begin{vmatrix} 1 & 1 & -2 \\ 1 & 1 & 4 \\ -4 & 1 & 7 \end{vmatrix}$

By $R_2 - R_1$ and $R_4 + 4R_1$

$(-3)\begin{vmatrix} 1 & 2 & -2 \\ 0 & 0 & 6 \\ 0 & 5 & -1 \end{vmatrix}$

Expanding from C_1

$(-3)\,1\begin{vmatrix} 0 & 6 \\ 5 & -1 \end{vmatrix} - 0 + 0$

$(-3)\,1(0-30) \Longrightarrow -3 \times -30$

$= 90 \neq 0$

So The matrix is non Singular.

Q.13 Find inverse of $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 3 \\ 2 & -4 & 1 \end{bmatrix}$ and Show that $A^{-1}A = I_3$

$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 3 \\ 2 & -4 & 1 \end{bmatrix}$

Cofactors of A are

$A_{11} = (-1)^{1+1}\begin{vmatrix} -1 & 3 \\ -4 & 1 \end{vmatrix} = 1(-1+12) = 11$

$A_{12} = (-1)^{1+2}\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = (-1)(1-6) = 5$

$A_{13} = (-1)^{1+3}\begin{vmatrix} 1 & -1 \\ 2 & -4 \end{vmatrix} = 1(-4+2) = -2$

$A_{21} = (-1)^{2+1}\begin{vmatrix} 1 & 0 \\ -4 & 1 \end{vmatrix} = (-1)(1-0) = -1$

$A_{22} = (-1)^{2+2}\begin{vmatrix} 2 & 0 \\ 2 & 1 \end{vmatrix} = 1(2-0) = 2$

$A_{23} = (-1)^{2+3}\begin{vmatrix} 2 & 1 \\ 2 & -4 \end{vmatrix} = (-1)(-8-2) = 10$

$A_{31} = (-1)^{3+1}\begin{vmatrix} 1 & 0 \\ -1 & 3 \end{vmatrix} = 1(3-0) = 3$

$A_{32} = (-1)^{3+2}\begin{vmatrix} 2 & 0 \\ 1 & 3 \end{vmatrix} = (-1)(6-0) = -6$

$A_{33} = (-1)^{3+3}\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = 1(-2-1) = -3$

Matrix of Cofactors of $A = \begin{bmatrix} 11 & 5 & -2 \\ -1 & 2 & 10 \\ 3 & -6 & -3 \end{bmatrix}$

$\text{Adj}A = (\text{Cofactor Matrix of } A)^t$

$\text{Adj}A = \begin{bmatrix} 11 & -1 & 3 \\ 5 & 2 & -6 \\ -2 & 10 & -3 \end{bmatrix}$

$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$

$|A| = (2)(11) + (1)(5) + (0)(-2)$

$|A| = 22 + 5 + 0 = 27$

$\therefore A^{-1} = \dfrac{\text{Adj}A}{|A|}$

$A^{-1} = \dfrac{1}{27}\begin{bmatrix} 11 & -1 & 3 \\ 5 & 2 & -6 \\ -2 & 10 & -3 \end{bmatrix}$ (Ans)

Now to Show

$A^{-1}A = I$

$$A^{-1}A=\frac{1}{27}\begin{bmatrix}11 & -1 & 3\\ 5 & 2 & -6\\ -2 & 10 & -3\end{bmatrix}\begin{bmatrix}2 & 1 & 0\\ 1 & -1 & 3\\ 2 & -4 & 1\end{bmatrix}$$

$$A^{-1}A=\frac{1}{27}\begin{bmatrix}22-1+6 & 11+1-2 & 0-3+0\\ 10+2-12 & 5-2+24 & 0+6-6\\ -4+10-6 & -2-10+12 & 0+30-3\end{bmatrix}$$

$$A^{-1}A=\frac{1}{27}\begin{bmatrix}27 & 0 & 0\\ 0 & 27 & 0\\ 0 & 0 & 27\end{bmatrix}$$

$$A^{-1}A=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$A^{-1}A = I_3$ (Proved)

Q.16 Verify that $(AB)^{-1} = B^{-1}A^{-1}$ if

(i) $A=\begin{bmatrix}1 & 2\\ -1 & 0\end{bmatrix}$ $\quad B=\begin{bmatrix}-3 & 1\\ 4 & -1\end{bmatrix}$

$|A|=\begin{vmatrix}1 & 2\\ -1 & 0\end{vmatrix}$ $\quad |B|=\begin{vmatrix}-3 & 1\\ 4 & -1\end{vmatrix}$

$|A| = 0+2$ $\quad |B| = (3-4)$

$|A| = 2$ $\quad |B| = -1$

$Adj A=\begin{bmatrix}0 & -2\\ 1 & 1\end{bmatrix}$ $\quad Adj B=\begin{bmatrix}-1 & -1\\ -4 & -3\end{bmatrix}$

$A^{-1}=\frac{Adj A}{|A|}$ $\quad B^{-1}=\frac{Adj B}{|B|}$

$A^{-1}=\frac{1}{2}\begin{bmatrix}0 & -2\\ 1 & 1\end{bmatrix}$ $\quad B^{-1}=-1\begin{bmatrix}-1 & -1\\ -4 & -3\end{bmatrix}$

$A^{-1}=\begin{bmatrix}0 & -1\\ 1/2 & 1/2\end{bmatrix}$ $\quad B^{-1}=\begin{bmatrix}1 & 1\\ 4 & 3\end{bmatrix}$

Now

$AB=\begin{bmatrix}1 & 2\\ -1 & 0\end{bmatrix}\begin{bmatrix}-3 & 1\\ 4 & -1\end{bmatrix}$

$AB=\begin{bmatrix}-3+8 & 1-2\\ 3+0 & -1+0\end{bmatrix}$

$AB=\begin{bmatrix}5 & -1\\ 3 & -1\end{bmatrix}$

$|AB| = (-5+3) = -2$

$Adj(AB)=\begin{bmatrix}-1 & 1\\ -3 & 5\end{bmatrix}$

$(AB)^{-1}=\frac{Adj(AB)}{|AB|}$

$(AB)^{-1}=\frac{1}{-2}\begin{bmatrix}-1 & 1\\ -3 & 5\end{bmatrix}$

$(AB)^{-1}=\begin{bmatrix}1/2 & -1/2\\ 3/2 & -5/2\end{bmatrix}$ ——— ①

Now $B^{-1}A^{-1}=\begin{bmatrix}1 & 1\\ 4 & 3\end{bmatrix}\begin{bmatrix}0 & -1\\ 1/2 & 1/2\end{bmatrix}$

$B^{-1}A^{-1}=\begin{bmatrix}0+1/2 & -1+1/2\\ 0+3/2 & -4+3/2\end{bmatrix}$

$B^{-1}A^{-1}=\begin{bmatrix}1/2 & -1/2\\ 3/2 & -5/2\end{bmatrix}$ ——— ②

From ① and ②, we see

$(AB)^{-1} = B^{-1}A^{-1}$

(Similarly do Part (ii) of this Question)

Q.15 Verify that $(AB)^t = B^tA^t$

$A=\begin{bmatrix}1 & -1 & 2\\ 0 & 3 & 1\end{bmatrix}$ $\quad B=\begin{bmatrix}1 & 1\\ 3 & 2\\ 0 & -1\end{bmatrix}$

$A^t=\begin{bmatrix}1 & 0\\ -1 & 3\\ 2 & 1\end{bmatrix}$ $\quad B^t=\begin{bmatrix}1 & 3 & 0\\ 1 & 2 & -1\end{bmatrix}$

$AB=\begin{bmatrix}1 & -1 & 2\\ 0 & 3 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ 3 & 2\\ 0 & -1\end{bmatrix}$

$AB=\begin{bmatrix}1-3+0 & 1-2-2\\ 0+9+0 & 0+6-1\end{bmatrix}$

$AB=\begin{bmatrix}-2 & -3\\ 9 & 5\end{bmatrix}$

$(AB)^t=\begin{bmatrix}-2 & 9\\ -3 & 5\end{bmatrix}$ ——— ①

$B^tA^t=\begin{bmatrix}1 & 3 & 0\\ 1 & 2 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\ -1 & 3\\ 2 & 1\end{bmatrix}$

$B^tA^t=\begin{bmatrix}1-3+0 & 0+9+0\\ 1-2-2 & 0+6-1\end{bmatrix}$

$B^tA^t=\begin{bmatrix}-2 & 9\\ -3 & 5\end{bmatrix}$ ——— ②

From ① and ② $\quad (AB)^t = B^tA^t$

Q.16 If $A=\begin{pmatrix}2 & -1\\3 & 1\end{pmatrix}$, then Show $(A^{-1})^t=(A^t)^{-1}$

$A=\begin{pmatrix}2 & -1\\-3 & 1\end{pmatrix}$

$|A| = 2+3 = 5$

$Adj A = \begin{pmatrix}1 & 1\\-3 & 2\end{pmatrix}$

$A^{-1} = \frac{Adj A}{|A|}$

$A^{-1} = \frac{1}{5}\begin{pmatrix}1 & 1\\-3 & 2\end{pmatrix}$

$A^{-1} = \begin{pmatrix}1/5 & 1/5\\-3/5 & 2/5\end{pmatrix}$

$(A^{-1})^t = \begin{pmatrix}1/5 & -3/5\\1/5 & 2/5\end{pmatrix}$ —— ①

$A^t = \begin{pmatrix}2 & 3\\-1 & 1\end{pmatrix}$

$|A^t| = 2+3 = 5$

$Adj(A^t) = \begin{pmatrix}1 & -3\\1 & 2\end{pmatrix}$

$(A^t)^{-1} = \frac{Adj(A^t)}{|A^t|}$

$(A^t)^{-1} = \frac{1}{5}\begin{pmatrix}1 & -3\\1 & 2\end{pmatrix}$

$(A^t)^{-1} = \begin{pmatrix}1/5 & -3/5\\1/5 & 2/5\end{pmatrix}$ —— ②

From ① and ②, we have

$(A^{-1})^t = (A^t)^{-1}$ (Proved)

Q.17 If A and B are non-Singular matrices then Show that

(i) $(AB)^{-1} = B^{-1}A^{-1}$

$(AB)^{-1} = B^{-1}A^{-1}$

Multiplying both Sides by (AB)

$(AB)(AB)^{-1} = (AB)\,B^{-1}A^{-1}$

$\because A\cdot A^{-1}=I$ Similarly $(AB)(AB)^{-1}=I$

$\therefore I = (AB)\,B^{-1}A^{-1}$

$I = A(BB^{-1})A^{-1}$ $\quad \because BB^{-1}=I$

$I = A(I)A^{-1}$ $\quad \because AI = A$

$I = AA^{-1}$

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$I = I$ (Proved)

(ii) $(A^{-1})^{-1} = A$

Multiplying both Sides by A^{-1}

$A^{-1}(A^{-1})^{-1} = A^{-1}A$

$\because AA^{-1}=A^{-1}A=I$ So $A^{-1}(A^{-1})^{-1}=I$

$\therefore I = I$ (Proved)

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Elementary Row Operation:

"The Operation used to find out the Solutions of the Systems of linear equations is called Elementary row operation."

Following Steps are used in Elementary row Operation.

(1) Interchanging two rows.

(2) Multiplying any row by any scalar number.

(3) Adding any multiple of one row into another row.

The notations used to represent the above Steps are

(i) $R_i \longleftrightarrow R_j$ or R_{ij} to interchange two rows.

(ii) $KR_i \longrightarrow R_i'$ to show scalar multiple of number.

(iii) $R_i + KR_j$ to represent the addition of any multiple row into another