Matrices And Determinants

Historical BackGround:

The word "Matrix" was first used by an English Mathematician James Sylvester and then in 1857 Arthur Cayley developed the theory of matrices and used them in linear transformation.

Tahir Mahmood
M.Sc. (Math)
Mob No: 0345-6510779

"Matrix"

"A rectangular array of numbers enclosed by a pair of brackets is called Matrix"

The horizonal lines of numbers are called Rows and Vertical lines of numbers are called Columns. The numbers present in rows or columns are called elements or entries. Such as

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

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Note:-
(i) Plural of matrix is Matrices.
(ii) The elements are denoted by small letters of alphabets while the name of matrices are denoted by Capital (Block) letters.

"Order of a Matrix"

If a Matrix has m rows and n columns then order of the matrix is defined as:

Order of Matrix $= m \times n$

Note: $m\times n$ does not mean the multiplication of m and n.

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Kinds Of Matrices:-

There are some types of Matrices.

(i) Row Matrix:-

"The matrix which has just one row is called Row matrix."

$m = 1$

$$A = \begin{bmatrix} 2 & 3 & 4 & 5 \end{bmatrix}$$

order $= 1\times n$

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(ii) Column Matrix:-

The matrix which has just only one column is called a Column Matrix.

$n = 1$

$$B = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

order $= m\times 1$

(iii) Square Matrix:

"The matrix in which no. of rows and columns are equal is known as Square matrix."

If $m = n$ then matrix is Square.

$$A = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

(iv) Rectangular Matrix: $(m \neq n)$

"The matrix in which no. of rows and Columns are not equal is called a rectangular matrix"

If $m \neq n$ then matrix is rectangular.

$$A = \begin{pmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 5 \\ 3 & 6 \\ 4 & 7 \end{pmatrix}$$

(v) Null or Zero Matrix:-

"The matrix in which every element is zero is called a null or zero matrix." It is denoted by $O_{m\times n}$

$$O_{2\times1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad O_{1\times1} = [0] \quad O_{1\times2} = [0 \ 0]$$

are the examples of null matrices.

(vi) Diagonal Matrix:-

Let a square matrix

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix}$$

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Then the entries a, f, k, p form a diagonal called Principal diagonal while the other diagonal having entries d, g, j, m is known as secondary diagonal "The matrix in which the elements other than principal diagonal are zero is called a Diagonal Matrix."

$$[5], \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Note: Diagonal Matrix is always a square matrix.

(vii) Scalar Matrix:-

"The diagonal Matrix in which the elements of a principal diagonal are identical (ایک جیسی) is called Scalar Matrix."

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

(viii) Unit or Identity Matrix:

"The diagonal matrix in which the elements of principal diagonal are 1 is called an Identity or Unit Matrix." It is denoted by $I_{m\times n}$ where $m\times n$ is the order of the matrix.

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$$[1], \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

are all unit or Identity matrices.

(ix) Equal Matrices:

"Two or more than two matrices are said to be Equal if and if only their orders are same and their Corresponding elements are same."

e.g.

$$A=\begin{bmatrix} 2 & 5 \\ 7 & 9 \end{bmatrix} \quad B=\begin{bmatrix} 4-2 & 3+2 \\ 7\times 1 & \frac{18}{2} \end{bmatrix}$$

A and B are equal matrices having same order 2×2 and Corresponding elements are also same.

(x) Negative of a Matrix:

"The matrix $-A$ is called the negative matrix of A if $-A$ has opposite sign for every entry."

e.g $A=\begin{bmatrix} -3 & 5 \\ 7 & -9 \end{bmatrix}$, $-A=\begin{bmatrix} 3 & -5 \\ -7 & 9 \end{bmatrix}$

(xi) Transpose Of a Matrix:-

"The matrix which is obtained by changing the rows of a matrix into columns or the columns of a matrix into rows is called transpose matrix."

If A is a matrix of $m\times n$ order then A transpose will be of $n\times m$ order and can be denoted as

$$A^t, \; A', \; \overline{A}$$

If $A=\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ then $A^t=\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$

(xii) Symmetric Matrix:

"The square matrix A is said to be Symmetric Matrix if $A^t = A$."

e.g. $A=\begin{bmatrix} a & b \\ b & a \end{bmatrix}$ $\quad A^t=\begin{bmatrix} a & b \\ b & a \end{bmatrix}$

$\because A^t = A$ so A is Symmetric Matrix.

(xiii) Skew-Symmetric Matrix:

"The square matrix A is said to be a skew symmetric Matrix if $A^t = -A$"

e.g. $A=\begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix}$ $\quad A^t=\begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$

$A^t = -A$ so A is Skew Symmetric.

OPERATION OF MATRICES:

(1) Addition Of Matrices:

"Two or more than two matrices are said to be Conformable for addition if they are of same order."

(i) To add two or more than two matrices we add their Corresponding elements.

(ii) The order of $(A+B)$ is the order of A as well as B.

Additive Identity Matrix:

"The matrix which does not change the elements of a matrix when added in the matrix is called Additive Identity Matrix."

Null matrix is called Additive Identity matrix.

If $A = \begin{pmatrix} 2 & 4 \\ 7 & -5 \end{pmatrix}$ $O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

then $A + O = O + A = A$

Note: The additive Identity matrix is used of the order which is also the order of the Given matrix.

Additive Inverse Matrix:

"The matrix which gives the Additive Identity matrix (Null Matrix) when added to another matrix is called Additive inverse of the Given matrix."

"In fact the Additive inverse of a matrix is the negative of the given matrix."

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$A = \begin{pmatrix} 2 & -3 \\ -4 & 5 \end{pmatrix}$ $-A = \begin{pmatrix} -2 & 3 \\ 4 & -5 \end{pmatrix}$

$A + (-A) = (-A) + A = O$

Properties of Matrix Addition:

(i) Commutative Property:

$A + B = B + A$

(ii) Associative Property:

$A + (B + C) = (A + B) + C$

(iii) Additive Identity Property:

$A + O = O + A = A$

(iv) Additive Inverse Property:

$A + (-A) = (-A) + A = O$

(2) Subtraction Of Matrices:

"Two or more than two matrices can be Subtracted if and if only they have same order."

Note: (i) The order of $(A-B)$ will be the order of A as well as B

(ii) The Corresponding elements are subtracted to subtract two matrices.

There are no Commutative or Associative Properties exist for Subtraction.

Scalar Multiplication:

"To multiply a matrix by any real number (Scalar Number) is called Scalar multiplication."

If $A = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$ then $KA = \begin{pmatrix} 5K & 6K \\ 7K & 8K \end{pmatrix}$

where K is any Scalar or real number.

(i) The order of KA = The order of A

(ii) $A + A + A + \cdots$ K terms $= KA$

i.e. $5A = A + A + A + A + A$

TAHIR MAHMOOD

(5)

(3) Multiplication Of Matrices:-

"The two matrices are said to be Conformable for multiplication if the no. of Columns of first matrix are equal to the no. of Rows of the Second matrix."

To multiply two matrices, we form the rows of first matrix while the Columns of 2nd Matrix.

"And the method of multiplication of two matrices is called "Row by Column Multiplication method."

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$

$\because$ No. of Columns of A = No. of rows of B

$\therefore$ AB is possible and if $O(A) = m \times n$ and $O(B) = n \times p$ then $O(AB) = m \times p$

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix}$$

$$= \begin{bmatrix} ap + br & aq + bs \\ pc + dr & cq + ds \end{bmatrix}$$

Properties Of Multiplication

(i) Commutative Property:-

Usually Commutative Property does not exist in Multiplication of two matrices i.e. $AB \neq BA$

But Some times it is possible then $AB = BA$

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(ii) Associative Property.

$A(BC) = (AB)C$

(iii) Multiplicative Identity Property.

$AI = IA = A$

(iv) Multiplicative Inverse Property.

$AA^{-1} = A^{-1}A = I$ for $|A| \neq 0$

Multiplicative Identity Matrix:

"The matrix which does not change the elements of the matrix when multiplied by a matrix is called Multiplicative identity matrix."

Unit matrix is identity matrix.

If A is matrix then

$AI = IA = A$

Note: The identity matrix is used of that order for which multiplication is Conformable.

Multiplicative Inverse Matrix:

"The matrix which gives the Multiplicative Identity (Unit) Matrix when multiplied by any matrix is called Multiplicative inverse of the given matrix." It is denoted by A^{-1} for the given matrix A.

$AA^{-1} = A^{-1}A = I$ for $|A| \neq 0$

Determinent of a matrix:-

"The value of a matrix in a real number is called the deter minant of the given matrix."

If A is the given matrix then deter minant is denoted by

$|A|$ or $\det A$

For 2x2 Matrices

$|A| = ad - bc$ for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Singular Matrix:-

"The matrix whose deter-minant equals to zero is called Singular or Non-Invertible matrix."

If $|A| = 0$ then A is Singular Matrix or Non Invertible Matrix.

e.g. $A = \begin{pmatrix} 9 & 3 \\ 6 & 2 \end{pmatrix}$

$|A| = 18 - 18 = 0$

A is Singular matrix.

Non-Singular Matrix:-

"The matrix whose determinant does not equal to zero is called Non Singular or Invertible matrix."

$\therefore$ If $|A| \neq 0$ then A is called Non Singular or Invertible matrix.

Adjoint of 2x2 Matrix:-

"In a square matrix, by changing the elements of principal diagonal and changing the signs of a secondary diagonal, we get the new matrix called the Adjoint of the given matrix."

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

then Adjoint A is denoted by $Adj A$ and defined by

$$Adj\, A = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Note:-

"Determinant was introduced by Seki Kuwa and Leibniz."

Multiplicative Inverse:

We have defined it already. Now A^{-1} can be found using the given formula.

$$A^{-1} = \frac{Adj A}{|A|} \quad \text{where } |A| \neq 0$$

If $|A| = 0$ then A^{-1} does not exist for the given matrix A.

Also

$AA^{-1} = A^{-1}A = I$ does not possible for $|A| = 0$

Tahir Mahmood
M.Sc. (Math)
Mob No: 0345-651077..
TAHIR
Tahir Mahmood
M.Sc. (Math)

Simultaneous Equations:

"The equations which have common solution are called Simultaneous Equations."

Such as
$$ax + by = p$$
$$cx + dy = q$$

Exercise 3.1

Q.1 $A = \begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix}$ $\quad B = \begin{pmatrix} 1 & 7 \\ 6 & 4 \end{pmatrix}$

(i) $4A - 3A = A$

$LHS = 4A - 3A$

$= 4\begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix} - 3\begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix}$

$= \begin{pmatrix} 8 & 12 \\ 4 & 20 \end{pmatrix} - \begin{pmatrix} 6 & 9 \\ 3 & 15 \end{pmatrix}$

$= \begin{pmatrix} 8-6 & 12-9 \\ 4-3 & 20-15 \end{pmatrix}$

$= \begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix} = A$

So $4A - 3A = A$ (Proved)

(ii) $3B - 3A = 3(B - A)$

$LHS = 3B - 3A$

$= 3\begin{pmatrix} 1 & 7 \\ 6 & 4 \end{pmatrix} - 3\begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix}$

$= \begin{pmatrix} 3 & 21 \\ 18 & 12 \end{pmatrix} - \begin{pmatrix} 6 & 9 \\ 3 & 15 \end{pmatrix}$

$= \begin{pmatrix} 3-6 & 21-9 \\ 18-3 & 12-15 \end{pmatrix}$

$= \begin{pmatrix} -3 & 12 \\ 15 & -3 \end{pmatrix}$

$RHS = 3(B - A)$

$= 3\left(\begin{pmatrix} 1 & 7 \\ 6 & 4 \end{pmatrix} - \begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix}\right)$

$= 3\begin{pmatrix} 1-2 & 7-3 \\ 6-1 & 4-5 \end{pmatrix}$

$= 3\begin{pmatrix} -1 & 4 \\ 5 & -1 \end{pmatrix}$

$= \begin{pmatrix} -3 & 12 \\ 15 & -3 \end{pmatrix}$

$\because LHS = RHS$

$3(B - A) = 3B - 3A$ (Proved)

Q.2 $A = \begin{pmatrix} i & 0 \\ 1 & -i \end{pmatrix}$

Prove $A^4 = I_{2\times 2}$

$A^2 = A \times A$

$A^2 = \begin{pmatrix} i & 0 \\ 1 & -i \end{pmatrix}\begin{pmatrix} i & 0 \\ 1 & -i \end{pmatrix}$

$A^2 = \begin{pmatrix} i\times i + 0\times 1 & i\times 0 + 0\times -i \\ 1\times i - i\times 1 & 1\times 0 + i\times i \end{pmatrix}$

$A^2 = \begin{pmatrix} i^2 & 0 \\ 0 & i^2 \end{pmatrix}$

$A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad \because i^2 = -1$

Now $A^4 = A^2 \times A^2$

$A^4 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$A^4 = \begin{bmatrix} (-1\times -1)+(0+0) & (-1\times 0)+(0\times -1) \\ (0\times -1)+(-1\times 0) & (0\times 0)+(-1\times -1) \end{bmatrix}$

$A^4 = \begin{bmatrix} 1+0 & 0+0 \\ 0+0 & 0+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A^4 = I_2$ (Proved)

* If $O(A) = m \times n$ and $O(B) = n \times q$ then $O(AB) = m \times q$

C.S.S. Academy
Tahir Mahmood M.Sc. (Math) Mob No: 0345-6510779
Tahir Mahmood
TAHIR

Q.3 Find x, y if

(i) $\begin{bmatrix} x+3 & 1 \\ -3 & 3y-4 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$

Being Equal Matrices their Corresponding elements must be Equal So

$x+3 = 2 \quad \wedge \quad 3y-4 = 2$

$x = 2-3 \quad \wedge \quad 3y = 2+4$

$x = 1 \quad \wedge \quad y = \frac{6}{3} = 2$

$x = 1 \quad \wedge \quad y = 2$

(ii) $\begin{bmatrix} x+3 & 1 \\ -3 & 3y-4 \end{bmatrix} = \begin{bmatrix} y & 1 \\ -3 & 2x \end{bmatrix}$

Being Equal Matrices their Corresponding elements must be Equal. So.

$x+3 = y \quad \wedge \quad 3y-4 = 2x$

$x+3 = y \quad \wedge \quad 2x-3y = -4$

Solving these Equations

$2x-3y = -4$

$2x-3(x+3) = -4 \quad (\because x+3 = y)$

$2x-3x-9 = -4$

$-x = 9-4$

$-x = 5$

$x = -5$

$y = x+3$

$y = -5+3$

$y = -2$

So

$x = -5 \qquad y = -2$

Q.4 $A = \begin{bmatrix} -1 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 3 & 2 \\ 1 & -1 & 2 \end{bmatrix}$

(i) $4A-3B = ?$

$4A-3B = 4\begin{bmatrix} -1 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix} - 3\begin{bmatrix} 0 & 3 & 2 \\ 1 & -1 & 2 \end{bmatrix}$

$4A-3B = \begin{bmatrix} -4 & 8 & 12 \\ 4 & 0 & 8 \end{bmatrix} - \begin{bmatrix} 0 & 9 & 6 \\ 3 & -3 & 6 \end{bmatrix}$

$4A-3B = \begin{bmatrix} -4-0 & 8-9 & 12-6 \\ 4-3 & 0+3 & 8-6 \end{bmatrix}$

$4A-3B = \begin{bmatrix} -4 & -1 & 6 \\ 1 & 3 & 2 \end{bmatrix}$ Ans.

(ii) $A+3(B-A) = ?$

$A+3(B-A) = \begin{bmatrix} -1 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix} + 3\left(\begin{bmatrix} 0 & 3 & 2 \\ 1 & -1 & 2 \end{bmatrix} - \begin{bmatrix} -1 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix}\right)$

$A+3(B-A) = \begin{bmatrix} -1 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix} + 3\begin{bmatrix} 0+1 & 3-2 & 2-3 \\ 1-1 & -1-0 & 2-2 \end{bmatrix}$

$A+3(B-A) = \begin{bmatrix} -1 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix} + 3\begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 0 \end{bmatrix}$

$A+3(B-A) = \begin{bmatrix} -1 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 3 & -3 \\ 0 & -3 & 0 \end{bmatrix}$

$A+3(B-A) = \begin{bmatrix} -1+3 & 2+3 & 3-3 \\ 1+0 & 0-3 & 2+0 \end{bmatrix}$

$A+3(B-A) = \begin{bmatrix} 2 & 5 & 0 \\ 1 & -3 & 2 \end{bmatrix}$ Ans.

Q.5 Find x and y if

$\begin{bmatrix} 2 & 0 & x \\ 1 & y & 3 \end{bmatrix} + 2\begin{bmatrix} 1 & x & y \\ 0 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 4 & -2 & 3 \\ 1 & 6 & 1 \end{bmatrix}$

Sol:- $\begin{bmatrix} 2 & 0 & x \\ 1 & y & 3 \end{bmatrix} + \begin{bmatrix} 2 & 2x & 2y \\ 0 & 4 & -2 \end{bmatrix} = \begin{bmatrix} 4 & -2 & 3 \\ 1 & 6 & 1 \end{bmatrix}$

$= \begin{bmatrix} 2+2 & 0+2x & x+2y \\ 1+0 & y+4 & 3-2 \end{bmatrix} = \begin{bmatrix} 4 & -2 & 3 \\ 1 & 6 & 1 \end{bmatrix}$

$= \begin{bmatrix} 4 & 2x & x+2y \\ 1 & y+4 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -2 & 3 \\ 1 & 6 & 1 \end{bmatrix}$

Being Equal Matrices Corresponding Elements must be Equal, So

$2x = -2$ —— (1) $\quad x+2y = 3$ —— (2)

$y+4 = 6$ —— (3)

From (1) $\quad x = -1$

Putting in (2) $\quad -1+2y = 3$

$2y = 3+1 \Rightarrow y = \frac{4}{2} = 2$

$\therefore x = -1$ and $y = 2$ Ans.

Tahir Mahmood
M.Sc. (Math)
Mob No: 0345-6510779
TAHIR

Q.6 If $A=[a_{ij}]_{3\times3}$ then show that:

(i) $\lambda(\mu A)=(\lambda\mu)A$

$\because [a_{ij}]_{3\times3}=A=\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}$

$LHS=\lambda\left[\mu\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}\right]$

$=\lambda\begin{pmatrix}\mu a_{11}&\mu a_{12}&\mu a_{13}\\\mu a_{21}&\mu a_{22}&\mu a_{23}\\\mu a_{31}&\mu a_{232}&\mu a_{33}\end{pmatrix}$

$=\begin{pmatrix}(\lambda\mu)a_{11}&(\lambda\mu)a_{12}&(\lambda\mu)a_{13}\\(\lambda\mu)a_{21}&(\lambda\mu)a_{22}&(\lambda\mu)a_{23}\\(\lambda\mu)a_{31}&(\lambda\mu)a_{32}&(\lambda\mu)a_{33}\end{pmatrix}$

$=\lambda\mu$ is a scalar multiplier so

$=(\lambda\mu)A$ (Proved).

(ii) $(\lambda+\mu)A=\lambda A+\mu A$

$RHS=\lambda A+\mu A$

$=\lambda\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}+\mu\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}$

$=\begin{bmatrix}\lambda a_{11}&\lambda a_{12}&\lambda a_{13}\\\lambda a_{21}&\lambda a_{22}&\lambda a_{23}\\\lambda a_{31}&\lambda a_{32}&\lambda a_{33}\end{bmatrix}+\begin{bmatrix}\mu a_{11}&\mu a_{12}&\mu a_{13}\\\mu a_{21}&\mu a_{22}&\mu a_{23}\\\mu a_{31}&\mu a_{32}&\mu a_{33}\end{bmatrix}$

$=\begin{bmatrix}\lambda a_{11}+\mu a_{11}&\lambda a_{12}+\mu a_{12}&\lambda a_{13}+\mu a_{13}\\\lambda a_{21}+\mu a_{21}&\lambda a_{22}+\mu a_{22}&\lambda a_{23}+\mu a_{23}\\\lambda a_{31}+\mu a_{31}&\lambda a_{32}+\mu a_{32}&\lambda a_{33}+\mu a_{33}\end{bmatrix}$

$=\begin{bmatrix}(\lambda+\mu)a_{11}&(\lambda+\mu)a_{12}&(\lambda+\mu)a_{13}\\(\lambda+\mu)a_{21}&(\lambda+\mu)a_{22}&(\lambda+\mu)a_{23}\\(\lambda+\mu)a_{31}&(\lambda+\mu)a_{32}&(\lambda+\mu)a_{33}\end{bmatrix}$

$=(\lambda+\mu)$ is scalar so

$=(\lambda+\mu)\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}$

$=(\lambda+\mu)A$ (Proved)

(iii) $\lambda A-A=(\lambda-1)A$

$LHS=\lambda A-A$

$=\lambda\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix}-\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix}$

$=\begin{bmatrix}\lambda a_{11}&\lambda a_{12}&\lambda a_{13}\\\lambda a_{21}&\lambda a_{22}&\lambda a_{23}\\\lambda a_{31}&\lambda a_{32}&\lambda a_{33}\end{bmatrix}-\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix}$

$=\begin{bmatrix}\lambda a_{11}-a_{11}&\lambda a_{12}-a_{12}&\lambda a_{13}-a_{13}\\\lambda a_{21}-a_{21}&\lambda a_{22}-a_{22}&\lambda a_{23}-a_{23}\\\lambda a_{31}-a_{31}&\lambda a_{32}-a_{32}&\lambda a_{33}-a_{33}\end{bmatrix}$

$=\begin{bmatrix}(\lambda-1)a_{11}&(\lambda-1)a_{12}&(\lambda-1)a_{13}\\(\lambda-1)a_{21}&(\lambda-1)a_{22}&(\lambda-1)a_{23}\\(\lambda-1)a_{31}&(\lambda-1)a_{32}&(\lambda-1)a_{33}\end{bmatrix}$

$(\lambda-1)$ is scalar so

$=(\lambda-1)\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}$

$=(\lambda-1)A$ (Proved)

Tahir Mahmood
M.Sc. (Math)
Mob No: 0343-6510779

Q.7 If $A=[a_{ij}]_{2\times3}$ and $B=[b_{ij}]_{2\times3}$

Show that $\lambda(A+B)=\lambda A+\lambda B$

Proof: $RHS=\lambda A+\lambda B$

$=\lambda\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\end{bmatrix}+\lambda\begin{bmatrix}b_{11}&b_{12}&b_{13}\\b_{21}&b_{22}&b_{23}\end{bmatrix}$

$=\lambda\left(\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\end{bmatrix}+\begin{bmatrix}b_{11}&b_{12}&b_{13}\\b_{21}&b_{22}&b_{23}\end{bmatrix}\right)$

$=\lambda(A+B)$ (Proved).

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Q.8 If $A=\begin{bmatrix}1&2\\a&b\end{bmatrix}$ & $A^2=\begin{bmatrix}0&0\\0&0\end{bmatrix}$ find a, b

$\because A=\begin{bmatrix}1&2\\a&b\end{bmatrix}$

$A^2=A\times A$

$A^2=\begin{bmatrix}1&2\\a&b\end{bmatrix}\begin{bmatrix}1&2\\a&b\end{bmatrix}$

$A^2=\begin{pmatrix}1+2a&2+2b\\a+ab&2a+b^2\end{pmatrix}$ ——— (1)

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But $A^2 = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$ —— (2)

Comparing (1) and (2).

$$\begin{pmatrix}1+2a & 2+2b\\ a+ab & 2a+b^2\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

Being Equal Matrices their Corresponding Elements are also Equal.

$1+2a=0 \quad \wedge \quad 2+2b=0$

$2a=-1 \quad \wedge \quad 2b=-2$

$a=-\frac{1}{2} \quad \wedge \quad b=\frac{-2}{2}$

$a=-\frac{1}{2} \quad \wedge \quad b=-1$ Ans.

Q.9 If $A=\begin{bmatrix}1 & -1\\ a & b\end{bmatrix}$ and $A^2=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ find a, b

$\because A^2 = A\times A$

$$A^2 = \begin{pmatrix}1 & -1\\ a & b\end{pmatrix}\begin{pmatrix}1 & -1\\ a & b\end{pmatrix}$$

$$A^2 = \begin{bmatrix}1-a & -1-b\\ a+ab & -a+b^2\end{bmatrix} \text{ —— (1)}$$

But $A^2 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ —— (2)

Comparing (1) and (2)

$$\begin{pmatrix}1-a & -1-b\\ a+ab & -a+b^2\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

Being Equal matrices, Corresponding elements must be Equal.

$\because 1-a=1 \quad \wedge \quad -1-b=0$

$a=1-1 \quad \wedge \quad b=-1$

$a=0 \quad \wedge \quad b=1$

So $a=0$, $b=1$ Ans.

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Q.10 Show that $(A+B)^t = A^t + B^t$

$$A=\begin{bmatrix}1 & -1 & 2\\ 0 & 3 & 1\end{bmatrix} \qquad B=\begin{pmatrix}2 & 3 & 0\\ 1 & 2 & -1\end{pmatrix}$$

LHS $= (A+B)$

$$(A+B) = \begin{bmatrix}1 & -1 & 2\\ 0 & 3 & 1\end{bmatrix} + \begin{bmatrix}2 & 3 & 0\\ 1 & 2 & -1\end{bmatrix}$$

$$(A+B) = \begin{bmatrix}1+2 & -1+3 & 2+0\\ 0+1 & 3+2 & 1+(-1)\end{bmatrix}$$

$$(A+B) = \begin{bmatrix}3 & 2 & 2\\ 1 & 5 & 0\end{bmatrix}$$

$$(A+B)^t = \begin{bmatrix}3 & 1\\ 2 & 5\\ 2 & 0\end{bmatrix} \text{ —— (1)}$$

RHS $= A^t + B^t \quad \because A^t = \begin{pmatrix}1 & 0\\ -1 & 3\\ 2 & 1\end{pmatrix} \quad B^t = \begin{pmatrix}2 & 1\\ 3 & 2\\ 0 & -1\end{pmatrix}$

$$A^t+B^t = \begin{bmatrix}1 & 0\\ -1 & 3\\ 2 & 1\end{bmatrix} + \begin{bmatrix}2 & 1\\ 3 & 2\\ 0 & -1\end{bmatrix}$$

$$A^t+B^t = \begin{bmatrix}1+2 & 0+1\\ -1+3 & 3+2\\ 2+0 & 1+(-1)\end{bmatrix}$$

$$A^t+B^t = \begin{bmatrix}3 & 1\\ 2 & 5\\ 2 & 0\end{bmatrix} \text{ —— (2)}$$

From (1) and (2)

LHS $=$ RHS $\Rightarrow (A+B)^t = A^t + B^t$

Q.11 Find A^3 if $A = \begin{bmatrix}1 & 1 & 3\\ 5 & 2 & 6\\ -2 & -1 & -3\end{bmatrix}$

$\because A^2 = A\times A$

$$A^2 = \begin{bmatrix}1 & 1 & 3\\ 5 & 2 & 6\\ -2 & -1 & -3\end{bmatrix}\begin{bmatrix}1 & 1 & 3\\ 5 & 2 & 6\\ -2 & -1 & -3\end{bmatrix}$$

$$A^2 = \begin{bmatrix}1+5+(-6) & 1+2+(-3) & 3+6+(-9)\\ 5+10+(-12) & 5+4+(-6) & 15+12+(-18)\\ -2-5+6 & -2-2+3 & -6-6+9\end{bmatrix}$$

$$A^2 = \begin{bmatrix}0 & 0 & 0\\ 3 & 3 & 9\\ -1 & -1 & -3\end{bmatrix}$$

Now $A^3 = A^2 \times A$

$$A^3 = \begin{bmatrix}0 & 0 & 0\\ 3 & 3 & 9\\ -1 & -1 & -3\end{bmatrix}\begin{bmatrix}1 & 1 & 3\\ 5 & 2 & 6\\ -2 & -1 & -3\end{bmatrix}$$

$$A^3 = \begin{bmatrix}0+0+0 & 0+0+0 & 0+0+0\\ 3+15-18 & 3+6-9 & 9+18-27\\ -1-5+6 & -1-2+3 & -3-6+9\end{bmatrix}$$

$$A^3 = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$A^3 = O_{3\times 3}$ Ans.

Tahir Mahmood
M.Sc. (Math)
Mob No: 03...

Q.12 Find the matrix X if:

(i) $X\begin{bmatrix}5 & 2\\ -2 & 1\end{bmatrix} = \begin{bmatrix}-1 & 5\\ 12 & 3\end{bmatrix}$

Let $A = \begin{pmatrix}5 & 2\\ -2 & 1\end{pmatrix}$ $\quad B = \begin{pmatrix}-1 & 5\\ 12 & 3\end{pmatrix}$

$\because XA = B$

$XAA^{-1} = BA^{-1}$

$XI = BA^{-1} \Rightarrow X = BA^{-1}$

So $A^{-1} = \dfrac{Adj A}{|A|}$

But $|A| = \begin{vmatrix}5 & 2\\ -2 & 1\end{vmatrix} = 5+4 = 9$

$Adj A = \begin{pmatrix}1 & -2\\ 2 & 5\end{pmatrix}$

$\because A^{-1} = \frac{1}{9}\begin{pmatrix}1 & -2\\ 2 & 5\end{pmatrix}$

Now $X = BA^{-1}$

$X = \frac{1}{9}\begin{pmatrix}-1 & 5\\ 12 & 3\end{pmatrix}\begin{pmatrix}1 & -2\\ 2 & 5\end{pmatrix}$

$X = \frac{1}{9}\begin{pmatrix}-1+10 & 2+25\\ 12+6 & -24+15\end{pmatrix}$

$X = \frac{1}{9}\begin{pmatrix}9 & 27\\ 18 & -9\end{pmatrix}$

$X = \begin{bmatrix}1 & 3\\ 2 & -1\end{bmatrix}$ Ans.

(ii) $\begin{bmatrix}5 & 2\\ -2 & 1\end{bmatrix}X = \begin{bmatrix}2 & 1\\ 5 & 10\end{bmatrix}$

Let $A = \begin{pmatrix}5 & 2\\ -2 & 1\end{pmatrix}$ $\quad B = \begin{pmatrix}2 & 1\\ 5 & 10\end{pmatrix}$

$AX = B \Longrightarrow A^{-1}AX = A^{-1}B$

$IX = A^{-1}B \Longrightarrow X = A^{-1}B$

$|A| = \begin{vmatrix}5 & 2\\ -2 & 1\end{vmatrix} = 5+4 = 9$

$Adj A = \begin{pmatrix}1 & -2\\ 2 & 5\end{pmatrix}$

$A^{-1} = \dfrac{Adj A}{|A|} = \frac{1}{9}\begin{pmatrix}1 & -2\\ 2 & 5\end{pmatrix}$

$\because X = A^{-1}B = \frac{1}{9}\begin{pmatrix}1 & -2\\ 2 & 5\end{pmatrix}\begin{bmatrix}2 & 1\\ 5 & 10\end{bmatrix}$

$X = \frac{1}{9}\begin{pmatrix}2-10 & 1-20\\ 4+25 & 2+50\end{pmatrix}$

$X = \frac{1}{9}\begin{pmatrix}-8 & -19\\ 29 & 52\end{pmatrix}$ Ans.

Q.13 Find matrix "A" if

$\begin{bmatrix}5 & -1\\ 0 & 0\\ 3 & 1\end{bmatrix}A = \begin{bmatrix}3 & -7\\ 0 & 0\\ 7 & 2\end{bmatrix}$

Sol:- Let $A = \begin{bmatrix}a & b\\ c & d\end{bmatrix}$

$\begin{bmatrix}5 & -1\\ 0 & 0\\ 3 & 1\end{bmatrix}\begin{bmatrix}a & b\\ c & d\end{bmatrix} = \begin{bmatrix}3 & -7\\ 0 & 0\\ 7 & 2\end{bmatrix}$

$\begin{bmatrix}5a-c & 5b-d\\ 0+0 & 0+0\\ 3a+c & 3b+d\end{bmatrix} = \begin{bmatrix}3 & -7\\ 0 & 0\\ 7 & 2\end{bmatrix}$

$\Rightarrow 5a-c = 3$ — (1) $\qquad 5b-d = -7$ — (3)

$3a+c = 7$ — (2) $\qquad 3b+d = 2$ — (4)

(1)+(2) $\Rightarrow 8a = 10$ $\qquad$ (3)+(4) $\Rightarrow 8b = -5$

$\boxed{a = \frac{10}{8} = \frac{5}{4}}$ $\qquad \boxed{b = -\frac{5}{8}}$

(1) $\Rightarrow c = 5a-3$ $\qquad$ (3) $\Rightarrow d = 5b+7$

$= 5\left(\frac{5}{4}\right)-3 = \frac{25-12}{4}$ $\qquad d = 5\left(\frac{-5}{8}\right)+7$

$\boxed{c = \frac{13}{4}}$ $\qquad d = \frac{-25+56}{8}$

$\boxed{d = \frac{31}{8}}$

Thus $A = \begin{bmatrix}5/4 & -5/8\\ 13/4 & 31/8\end{bmatrix}$ Ans.

(ii) $\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}A = \begin{bmatrix}0 & -3 & 8\\ 3 & 3 & -7\end{bmatrix}$

Let $B = \begin{pmatrix}2 & -1\\ -1 & 1\end{pmatrix}$ $\quad C = \begin{pmatrix}0 & -3 & 8\\ 3 & 3 & -7\end{pmatrix}$

$\because BA = C \Longrightarrow B^{-1}BA = B^{-1}C$

$IA = B^{-1}C \Longrightarrow A = B^{-1}C$

$|B| = \begin{vmatrix}2 & -1\\ -1 & 2\end{vmatrix} = 4-1 = 3$

$Adj B = \begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}$

$B^{-1} = \frac{1}{3}\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}$

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$\because A = B^{-1}C = \frac{1}{3}\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}\begin{pmatrix}0 & -3 & 8\\ 3 & 3 & -7\end{pmatrix}$

$A = \frac{1}{3}\begin{pmatrix}0+3 & -6+3 & 16-7\\ 0+6 & -3+6 & 8-14\end{pmatrix}$

$A = \frac{1}{3}\begin{pmatrix}3 & -3 & 9\\ 6 & 3 & -6\end{pmatrix}$

$A = \begin{pmatrix}1 & -1 & 3\\ 2 & 1 & -2\end{pmatrix}$ Ans.

TAHIR MAHMOOD

Q.14 Show that

$$\begin{bmatrix} r\cos\phi & 0 & -\sin\phi \\ 0 & r & 0 \\ r\sin\phi & 0 & \cos\phi \end{bmatrix}\begin{bmatrix}\cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -r\sin\phi & 0 & r\cos\phi\end{bmatrix} = rI_3$$

$$LHS = \begin{bmatrix} r\cos\phi & 0 & -\sin\phi \\ 0 & r & 0 \\ r\sin\phi & 0 & \cos\phi \end{bmatrix}\begin{bmatrix}\cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -r\sin\phi & 0 & r\cos\phi\end{bmatrix}$$

$$= \begin{bmatrix} r\cos^2\phi+0+r\sin^2\phi & 0+0+0 & \cancel{r\sin\phi\cos\phi}+0-\cancel{r\sin\phi\cos\phi} \\ 0+0+0 & 0+r+0 & 0+0+0 \\ \cancel{r\sin\phi\cos\phi}+0-\cancel{r\sin\phi\cos\phi} & 0+0+0 & r\sin^2\phi+0+r\cos^2\phi \end{bmatrix}$$

$$= \begin{bmatrix} r(\sin^2\phi+\cos^2\phi) & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r(\sin^2\phi+\cos^2\phi) \end{bmatrix}$$

$$= \begin{bmatrix} r(1) & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r(1) \end{bmatrix} \quad (\because \sin^2\theta+\cos^2\theta=1)$$

$$= \begin{bmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \end{bmatrix}$$

Can also be written as

$$= \begin{bmatrix} 1\times r & 0\times r & 0\times r \\ 0\times r & 1\times r & 0\times r \\ 0\times r & 0\times r & 1\times r \end{bmatrix}$$

$\therefore$ r is Scalar So

$$r\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = r\, I_{3\times 3} \text{ (Proved).}$$

Some further Properties:-

(1) Distributive Property w.r.t addition.

$A(B+C) = AB+AC$ Left Dis. Property

$(A+B)C = AC+BC$ Right Dis. Property

(2) Scalar Multiplication Properties.

(i) Scalar Distributive Property

$c(A+B) = cA+cB$, $(c+d)A = cA+dA$

Exercise: 32

(1) If $A = [a_{ij}]_{3\times 4}$ Then Show that

(i) $I_3 A = A$

$LHS = I_3 A$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}+0+0 & a_{12}+0+0 & a_{13}+0+0 & a_{14}+0+0 \\ 0+a_{21}+0 & 0+a_{22}+0 & 0+a_{23}+0 & 0+0+a_{24} \\ 0+0+a_{31} & 0+0+a_{32} & 0+0+a_{33} & 0+0+a_{34} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = A$$

So LHS = RHS $\Rightarrow I_3A = A$ (Proved)

Tahir Mahmood
M.Sc. (Math)
Mob No: 0345-6510779

Tahir Mahmood
M.Sc. (Math)
Mob No: 0345-6510779

(ii) $AI_4 = A$

$LHS = AI_4$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}+0+0+0 & 0+a_{12}+0+0 & 0+0+a_{13}+0 & 0+0+0+a_{14} \\ a_{21}+0+0+0 & 0+a_{22}+0+0 & 0+0+a_{23}+0 & 0+0+0+a_{24} \\ a_{31}+0+0+0 & 0+a_{32}+0+0 & 0+0+a_{33}+0 & 0+0+0+a_{34} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

$= A$

So LHS = RHS.

Hence $AI_4 = A$ (Proved)

Tahir Mahmood
TAHIR