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M.Sc Math
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Operation:-

An operation is a rule which when applied upon one or more quantities provide another quantity.

e.g. $+, \times, \div, -, \sqrt{\ }$ etc.

Binary Operation:-

An operation $*: S \times S \to S$ is called binary operation if

$a, b \in S \Rightarrow a * b \in S$

Some well known binary operations are:

Addition	$+$
Multiplication	$\times$
Subtraction	$-$
Division	$\div$
Union	$\cup$
Intersection	$\cap$

Groupoid:-

Let $G \neq \phi$ and $*$ be a binary operation then $(G, *)$ is called Monoid if

$\forall\, a, b \in G \Rightarrow a * b \in G$

e.g Set of natural number under "+".

Semi Group:-

An ordered pair $(G, *)$ is called Semi group if

i) $\forall a, b \in G \Rightarrow a * b \in G$

ii) $\forall\, a, b, c \in G$

$a * (b * c) = (a * b) * c$

e.g Set of natural number under "+"

Set of Whole numbers under "+"

Monoid:-

An ordered pair $(G, *)$ is called Monoid if

i) $\forall\, a, b \in G \Rightarrow a * b \in G$

ii) $\forall\, a, b, c \in G$

$a * (b * c) = (a * b) * c$

iii) $\forall\, a \in G \quad \exists\, e \in G$

Suchthat $a * e = e * a = a$

e is identity of G under $*$.

Group:-

An ordered pair $(G, *)$ is called Group if

G_1: $\forall a, b \in G \Rightarrow a * b \in G$

G_2: $\forall\, a, b, c \in G$

$a * (b * c) = (a * b) * c$

G_3: $\forall\, a \in G \quad \exists\, e \in G$

Suchthat $a * e = e * a = a$

e is identity of G under "$*$"

G_4: $\forall\, a \in G \quad \exists\, a^{-1} \in G$

Suchthat $a * a^{-1} = a^{-1} * a = e$

a^{-1} is called inverse of a under "$*$"

e.g Set of real numbers under "+"

Abelian group:-

A group $(G, *)$ is called abelian group if $\forall\, a, b \in G$

$a * b = b * a$

It was introduced by N.H. Abel.

e.g

→ Group of real numbers under "+"

→ Group of non zero rational numbers under "$\cdot$"

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Ex: 2.8

Order of a group:-

The number of elements in a group G is called its order and is denoted by $|G|$ or $O(G)$.

Finite and Infinite Group:-

A group G is called finite if its order is finite otherwise G is called infinite group.

Theorem

In a group $(G,*)$, identity e is unique.

Proof:- Suppose e_1 and e_2 are two identities in G then

$e_1 * e_2 = e_1$ (e_2 as identity)

$e_1 * e_2 = e_2$ (e_1 as identity)

$\Rightarrow e_1 = e_2 = e$ (let)

$\Rightarrow$ Identity e is Unique.

2008

Theorem

In a group $(G,*)$, inverse of an element $a \in G$ is unique.

Proof:- Suppose a_1^{-1}, a_2^{-1} are two inverses of $a \in G$ then

$a * a_1^{-1} = e = a_1^{-1} * a$ —①

and $a * a_2^{-1} = e = a_2^{-1} * a$ —②

Now suppose

$a_1^{-1} * (a * a_2^{-1}) = (a_1^{-1} * a) * a_2^{-1}$

$\Rightarrow a_1^{-1} * (e) = e * a_2^{-1}$

$\Rightarrow a_1^{-1} = a_2^{-1}$ $\because a*e = e*a = a$

$\Rightarrow$ Inverse is unique in G.

Theorem:-

Solve (i) $a * x = b$

(ii) $x * a = b$

where $a, b \in G$ (G being Group)

Sol:- (i) Consider

$a * x = b$

$\because a, b \in G \Rightarrow a^{-1}, b^{-1} \in G$

$\Rightarrow a^{-1} * (a * x) = a^{-1} * b$

$\Rightarrow (a^{-1} * a) * x = a^{-1} * b$ (By Ass. Law)

$\Rightarrow e * x = a^{-1} * b$ $(\because a^{-1} * a = e)$

$\Rightarrow x = a^{-1} * b$ $(\because e * a = a)$

which is required Solution.

(ii) Consider

$x * a = b$

$\because a, b \in G \Rightarrow a^{-1}, b^{-1} \in G$

$\Rightarrow (x * a) * a^{-1} = b * a^{-1}$

$\Rightarrow x * (a * a^{-1}) = b * a^{-1}$ (by Ass. Law)

$\Rightarrow x * e = b * a^{-1}$ $(\because a * a^{-1} = e)$

$\Rightarrow x = b * a^{-1}$ $(\because a * e = a)$

which is required Solution.

Q. Show that set $G = \{1, \omega, \omega^2\}$ where $\omega^3 = 1$ is abelian group under multiplication.

Sol:- Consider $G = \{1, \omega, \omega^2\}$ where $\omega^3 = 1$

G_1: G is closed by table under "."

G_2: G is closed so is associative under "."

G_3: Clearly $1 \in G$ which is identity under "."

$\times$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

G_4: Clearly

$(1)(1) = 1 \Rightarrow (1)^{-1} = 1 \in G$

$(\omega)(\omega^2) = 1 \Rightarrow \omega^{-1} = \omega^2 \in G$

$(\omega^2)(\omega) = 1 \Rightarrow (\omega^2)^{-1} = \omega \in G$

$\Rightarrow$ G has inverse of each of its element.

G_5: G is abelian as table is Symmetric about diagonal.

$\Rightarrow$ G is abelian group under ".".

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Q. Show that set $S=\{1,-1,i,-i\}$ where $i^4=1$, is abelian group under multiplication.

Sol:- Consider $S=\{1,-1,i,-i\}$

$\times$	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	i
i	i	$-i$	-1	1
$-i$	$-i$	i	1	-1

From multiplication table

G_1: S is closed under "$\circ$"

G_2: S is associative as closed.

G_3: Clearly $1\in S$
which is multiplicative identity under "$\circ$"

G_4: Clearly

$1\cdot 1=1 \Rightarrow (1)^{-1}=1\in S$

$(-1)(-1)=1 \Rightarrow (-1)^{-1}=-1\in S$

$(i)(-i)=1 \Rightarrow i^{-1}=-i\in S$

$(-i)(i)=1 \Rightarrow (-i)^{-1}=i\in S$

$\Rightarrow$ S has multiplicative inverse of each of its element.

G_5: Clearly Table is Symmetric about diagonal
so S is abelian group.

2007

Q. Show that set Consisting the elements of the Form $a+\sqrt{3}b$ $(a,b\in Q)$ is an abelian group under addition.

Sol:- G_1: let $a+\sqrt{3}b,\ c+\sqrt{3}d \in G=\{a+\sqrt{3}b \mid a,b\in Q\}$

$(a+\sqrt{3}b)+(c+\sqrt{3}d)=(a+c)+\sqrt{3}(b+d)\in G$ as $a+c,\ b+d\in Q$.
G is closed under addition.

G_2: G is associative as it is closed under addition.

G_3: Clearly $0=0+\sqrt{3}(0)\in G \Rightarrow 0+\sqrt{3}(0)$ is identity under "+".

G_4: let $a+\sqrt{3}b\in G \Rightarrow -a-\sqrt{3}b\in G$ For $a,b\in Q\Rightarrow -a,-b\in Q$
Such that $(a+\sqrt{3}b)+(-a-\sqrt{3}b)=0=(-a-\sqrt{3}b)+(a+\sqrt{3}b)$
$\Rightarrow$ G has inverse of each of its element.

G_5: Clearly Q is abelian so G is abelian by inherited Property
$\Rightarrow (G,+)$ is abelian group.

"The End"