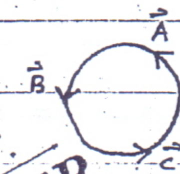
Scalar Triple Product:- (S.T.P)

Let $\vec{A}, \vec{B}, \vec{C}$ are three vectors then their scalar triple product is denoted and defined as

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = (\vec{A}\times\vec{B})\cdot\vec{C} = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

Charactristics:-

(i) The STP remains same if vectors are in cyclic order.



$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$

$(\vec{A}\times\vec{B})\cdot\vec{C} = (\vec{B}\times\vec{C})\cdot\vec{A} = (\vec{C}\times\vec{A})\cdot\vec{B}$

(ii) The STP change by sign if it's cyclic order is disturbed by one vector.

$\vec{A}\cdot(\vec{C}\times\vec{B}) = -\vec{A}\cdot(\vec{B}\times\vec{C})$ etc.

(iii) If $\vec{A}, \vec{B}, \vec{C}$ are Coplaner vectors then $\vec{A}\cdot(\vec{B}\times\vec{C}) = 0$

(iv) If Any two or more vectors are identical then STP = 0

(v) If Any one vector is a linear Combination of others then STP = 0

i.e. let $\vec{C} = \lambda\vec{A} + \mu\vec{B}$ so $\vec{A}\cdot(\vec{B}\times\vec{C}) = 0$

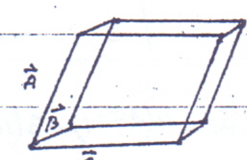
(vi) Geometrically STP represents the volume of parallelopiped having vectors as Conterminous edges.

Vol. of Parallelopiped $= \vec{A}\cdot(\vec{B}\times\vec{C})$

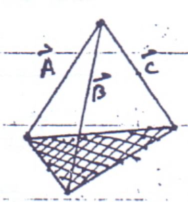
(vii) Volume of tetra hydron $= \frac{1}{6}\{\vec{A}\cdot(\vec{B}\times\vec{C})\}$

where $\vec{A}, \vec{B}, \vec{C}$ are Conterminous edges.

Parallelopiped.



Tetrahedron



Work done.

"The dot product of force and displacement is known as work done by the body under the applied Force."

$$W = \vec{F}\cdot\vec{d}$$

Torque or Moment of force.

The turning effect produce in a body due to applied Force is torque

$$\vec{\tau} = \vec{r}\times\vec{F}$$

where $\vec{r}$ is called moment arm.

Exercise 7.5

Q.1 (i) $\vec{u} = 3\hat{i}+2\hat{k}$, $\vec{v} = \hat{i}+2\hat{j}+\hat{k}$, $\vec{w} = -\hat{j}+4\hat{k}$

Vol. of ||piped $= \vec{u}\cdot(\vec{v}\times\vec{w})$

$$= \begin{vmatrix} 3 & 0 & 2 \\ 1 & 2 & 1 \\ 0 & -1 & 4 \end{vmatrix}$$

$= 3(8+1) - 1(0+2) + 0$

$= 27 - 2 = 25$ cubic units.

$\Rightarrow$ Vol. of ||piped $= 25$ cubic units

(Similarly Remaining parts).

Q.2 Verify $\vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{b}\cdot(\vec{c}\times\vec{a}) = \vec{c}\cdot(\vec{a}\times\vec{b})$

$\vec{a} = 3\hat{i}-\hat{j}+5\hat{k}$ $\quad \vec{b} = 4\hat{i}+3\hat{j}-2\hat{k}$ $\quad \vec{c} = 2\hat{i}+5\hat{j}+\hat{k}$

Soln: $\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} 3 & -1 & 5 \\ 4 & 3 & -2 \\ 2 & 5 & 1 \end{vmatrix}$

$\vec{a}\cdot(\vec{b}\times\vec{c}) = 3(3+10) + 1(4+4) + 5(20-6)$

$\vec{a}\cdot(\vec{b}\times\vec{c}) = 39+8+70 = 117$ —— (1)

$\vec{b}\cdot(\vec{c}\times\vec{a}) = \begin{vmatrix} 4 & 3 & -2 \\ 2 & 5 & 1 \\ 3 & -1 & 5 \end{vmatrix}$

$\vec{b}\cdot(\vec{c}\times\vec{a}) = 4(25+1) - 3(10-3) - 2(-2-15)$

$\vec{b}\cdot(\vec{c}\times\vec{a}) = 104 - 21 + 34 = 117$ —— (2)

$\vec{c}\cdot(\vec{a}\times\vec{b}) = \begin{vmatrix} 2 & 5 & 1 \\ 3 & -1 & 5 \\ 4 & 3 & -2 \end{vmatrix}$

$\vec{c}\cdot(\vec{a}\times\vec{b}) = 2(2-15) - 5(-6-20) + 1(9+4)$

$\vec{c}\cdot(\vec{a}\times\vec{b}) = -26 + 130 + 13 = 117$ —— (3)

From (1), (2), (3)

$\vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{b}\cdot(\vec{c}\times\vec{a}) = \vec{c}\cdot(\vec{a}\times\vec{b})$

Proved!

Q.3 let $\vec{a} = \hat{i}-2\hat{j}+3\hat{k}$, $\vec{b} = -2\hat{i}+3\hat{j}-4\hat{k}$

$\vec{c} = \hat{i}-3\hat{j}+5\hat{k}$

Consider: $\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} 1 & -2 & 3 \\ -2 & 3 & -4 \\ 1 & -3 & 5 \end{vmatrix}$

$\Rightarrow \vec{a}\cdot(\vec{b}\times\vec{c}) = 1(15-12) + 2(-10+4) + 3(6-3)$

$\Rightarrow \vec{a}\cdot(\vec{b}\times\vec{c}) = 3 - 12 + 9 = 0$

$\because \vec{a}\cdot(\vec{b}\times\vec{c}) = 0$ so

$\vec{a}, \vec{b}, \vec{c}$ are Coplaner vectors.

Q.4 Find $\alpha = ?$ Vectors are Coplaner.

(i) $\hat{i}-\hat{j}+\hat{k}$, $\hat{i}-2\hat{j}-3\hat{k}$, $3\hat{i}-\alpha\hat{j}+5\hat{k}$

$\because$ Vectors are Coplaner so

$$\begin{vmatrix} 1 & -1 & 1 \\ 1 & -2 & -3 \\ 3 & -\alpha & 5 \end{vmatrix} = 0$$

$1(-10-3\alpha) + 1(5+9) + 1(-\alpha+6) = 0$

$-10 - 3\alpha + 14 - \alpha + 6 = 0 \Rightarrow -4\alpha + 10 = 0$

$4\alpha = 10 \Rightarrow \boxed{\alpha = 5/2}$

(ii) $\hat{i}-2\alpha\hat{j}-\hat{k}$, $\hat{i}-\hat{j}+2\hat{k}$, $\alpha\hat{i}-\hat{j}+\hat{k}$

Take $\begin{vmatrix} 1 & -2\alpha & -1 \\ 1 & -1 & 2 \\ \alpha & -1 & 1 \end{vmatrix} = 0$

$1(-1+2) - 1(-2\alpha-1) + \alpha(-4\alpha-1) = 0$

$1 + 2\alpha + 1 - 4\alpha^2 - \alpha = 0$

$\Rightarrow 4\alpha^2 - \alpha - 2 = 0 \Rightarrow \alpha = \dfrac{1 \pm \sqrt{1+16}}{8}$

$\boxed{\alpha = \dfrac{1 \pm \sqrt{17}}{8}}$ Ans.

Tahir Mahmood
M.Sc. (Math)
Mob No: 0345-6510779

Q.5 (a) Find the Value of :

(i) $(2i \times 2j)\cdot\hat{k} = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

$(2i \times 2j)\cdot\hat{k} = 2(2-0) - 0 + 0$

$(2\hat{i} \times 2\hat{j})\cdot\hat{k} = 4$ Ans.

(ii) $3\hat{j}\cdot(\hat{k} \times \hat{i}) = \begin{vmatrix} 0 & 3 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix}$

$3\hat{j}\cdot(\hat{k}\times\hat{i}) = 0 - 0 + 1(3-0)$

$3\hat{j}\cdot(\hat{k}\times\hat{i}) = 3$ Ans.

(iii) $[\hat{k}\ \hat{i}\ \hat{j}] = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}$

$[\hat{k}\ \hat{i}\ \hat{j}] = 0 - 0 + 1(1-0)$

$[\hat{k}\ \hat{i}\ \hat{j}] = 1$ Ans.

(iv) $[\hat{i}\ \hat{i}\ \hat{k}] = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

$[\hat{i}\ \hat{i}\ \hat{k}] = 1(0-0) - 0 + 0 = 0$

$[\hat{i}\ \hat{i}\ \hat{k}] = 0$ Ans.

(b) let $\vec{u} = (u_1\hat{i} + u_2\hat{j} + u_3\hat{k})$

$\vec{v} = (v_1\hat{i} + v_2\hat{j} + v_3\hat{k})$

$\vec{w} = (\omega_1\hat{i} + \omega_2\hat{j} + \omega_3\hat{k})$

$\vec{u}\cdot(\vec{v}\times\vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ \omega_1 & \omega_2 & \omega_3 \end{vmatrix}$ —— ①

Now $\vec{v}\cdot(\vec{\omega}\times\vec{u}) = \begin{vmatrix} v_1 & v_2 & v_3 \\ \omega_1 & \omega_2 & \omega_3 \\ u_1 & u_2 & u_3 \end{vmatrix}$

$= -\begin{vmatrix} v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \\ \omega_1 & \omega_2 & \omega_3 \end{vmatrix}$ By $R_2 \leftrightarrow R_3$

$= (-1)^2\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ \omega_1 & \omega_2 & \omega_3 \end{vmatrix}$ By $R_1 \leftrightarrow R_2$

$\vec{v}\cdot(\vec{\omega}\times\vec{u}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ \omega_1 & \omega_2 & \omega_3 \end{vmatrix}$ —— ②

Now $\vec{\omega}\cdot(\vec{u}\times\vec{v}) = \begin{vmatrix} \omega_1 & \omega_2 & \omega_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$

$= (-1)\begin{vmatrix} u_1 & u_2 & u_3 \\ \omega_1 & \omega_2 & \omega_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$ By $R_1 \leftrightarrow R_2$

$= (-1)^2\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ \omega_1 & \omega_2 & \omega_3 \end{vmatrix}$ By $R_2 \leftrightarrow R_3$

$\vec{\omega}\cdot(\vec{u}\times\vec{v}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ \omega_1 & \omega_2 & \omega_3 \end{vmatrix}$ —— ③

Adding ①, ②, ③, we have

$\vec{u}\cdot(\vec{v}\times\vec{\omega}) + \vec{v}\cdot(\vec{\omega}\times\vec{u}) + \vec{\omega}\cdot(\vec{u}\times\vec{v}) = 3\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ \omega_1 & \omega_2 & \omega_3 \end{vmatrix}$

$\Rightarrow \vec{u}\cdot(\vec{v}\times\vec{\omega}) + \vec{v}\cdot(\vec{\omega}\times\vec{u}) + \vec{\omega}\cdot(\vec{u}\times\vec{v}) = 3\,\vec{u}\cdot(\vec{v}\times\vec{\omega})$

(Proved).

Challenge : Is it possible to evaluate $\vec{a}\times(\vec{b}\cdot\vec{c})$ and $\vec{a}\cdot(\vec{b}\cdot\vec{c})$?

If not give the reason :

Tahir Mahmood
M.Sc. (Math)
Mob No: 0345-6510779

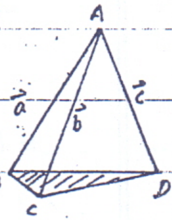
Tahir Mahmood
M.Sc. (Math)
Mob No: 0345-6510779

Q.6 Find Vol. of tetrahedron:

(i) let $A(0,1,2)$, $B(3,2,1)$, $C(1,2,1)$, $D(5,5,6)$

let $\vec{a} = \vec{AB} = (3-0)\hat{i}+(2-1)\hat{j}+(1-2)\hat{k}$

$\vec{a} = 3\hat{i}+\hat{j}-\hat{k}$



let $\vec{b} = \vec{AC} = (1-0)\hat{i}+(2-1)\hat{j}+(1-2)\hat{k}$

$\vec{b} = \hat{i}+\hat{j}-\hat{k}$

let $\vec{c} = \vec{AD} = (5-0)\hat{i}+(5-1)\hat{j}+(6-2)\hat{k}$

$\vec{c} = 5\hat{i}+4\hat{j}+4\hat{k}$

Vol. of Tetrahedron $= \frac{1}{6}\{\vec{a}\cdot(\vec{b}\times\vec{c})\}$

Vol. of $\triangle ABCD = \frac{1}{6}\begin{vmatrix} 3 & 1 & -1 \\ 1 & 1 & -1 \\ 5 & 4 & 4 \end{vmatrix}$

Vol. of $\triangle ABCD = \frac{1}{6}\{3(4+4)-1(4+5)-1(4-5)\}$

$= \frac{1}{6}\{24-9+1\} = \frac{16}{6} = \frac{8}{3}$

Vol. of $\triangle ABCD = 8/3$ cubic Unit.

(ii) Do yourself Similarly)

Q.7 $W = ?$ $\vec{F} = 4\hat{i}+3\hat{j}+5\hat{k}$

$\vec{d} = \vec{P_1P_2} = (2-3)\hat{i}+(4-1)\hat{j}+(6+2)\hat{k}$

$\vec{d} = -\hat{i}+3\hat{j}+8\hat{k}$

$\because W = \vec{F}\cdot\vec{d}$

$W = (4\hat{i}+3\hat{j}+5\hat{k})\cdot(-\hat{i}+3\hat{j}+8\hat{k})$

$W = -4+9+40 = 45$ unit Ans.

Q.8 let $\vec{F_1} = 4\hat{i}+\hat{j}-3\hat{k}$ & $\vec{F_2} = 3\hat{i}-\hat{j}-\hat{k}$

$\vec{d} = \vec{AB} = (5-1)\hat{i}+(4-2)\hat{j}+(1-3)\hat{k}$

$\vec{d} = 4\hat{i}+2\hat{j}-2\hat{k}$

$\vec{F} = \vec{F_1}+\vec{F_2} = 7\hat{i}+0\hat{j}-4\hat{k}$

Now $W = \vec{F}\cdot\vec{d}$

$W = (7\hat{i}-4\hat{k})\cdot(4\hat{i}+2\hat{j}-2\hat{k})$

$W = 28+0+8 = 36$ unit Ans.

Q.9 :- let $\vec{F_1} = 10\hat{i}-\hat{j}+11\hat{k}$

$\vec{F_2} = 4\hat{i}+5\hat{j}+9\hat{k}$. $\vec{F_3} = -2\hat{i}+\hat{j}-9\hat{k}$

$\vec{d} = \vec{AB} = (6-5)\hat{i}+(2+5)\hat{j}+(-2+7)\hat{k}$

$\vec{d} = \hat{i}+7\hat{j}+5\hat{k}$

$\vec{F} = \vec{F_1}+\vec{F_2}+\vec{F_3} = 12\hat{i}+5\hat{j}+11\hat{k}$

Now $W = \vec{F}\cdot\vec{d}$

$W = (12\hat{i}+5\hat{j}+11\hat{k})\cdot(\hat{i}+7\hat{j}+5\hat{k})$

$W = 12+35+55 = 102$ Unit Ans.

Q.10 let $|F| = 6$ & $\vec{u} = 2\hat{i}-2\hat{j}+\hat{k}$

$|\vec{u}| = \sqrt{4+4+1} = 3 \Rightarrow \hat{u} = \frac{2\hat{i}-2\hat{j}+\hat{k}}{3}$

$\because \vec{F} \parallel \vec{u}$ so $\hat{F} = \hat{u}$

Thus $\vec{F} = |F|\hat{u} = 6\left(\frac{2\hat{i}-2\hat{j}+\hat{k}}{3}\right)$

$\vec{F} = 4\hat{i}-4\hat{j}+2\hat{k}$

Now $\vec{d} = (5-1)\hat{i}+(3-2)\hat{j}+(7-3)\hat{k}$

$\Rightarrow \vec{d} = 4\hat{i}+\hat{j}+4\hat{k}$

$\Rightarrow W = \vec{F}\cdot\vec{d}$

$W = (4\hat{i}-4\hat{j}+2\hat{k})\cdot(4\hat{i}+\hat{j}+4\hat{k})$

$W = 16-4+8 = 20$ unit

$W = 20$ unit Ans.

Contact No: 0345 6510779
Tahir Mahmood
M.Sc. (Math)
Mob No: 0345-6510779

Tahir Mahmood
M.Sc. (Math)
Mob No: 0345-6510779

Q.11:- $\vec{F} = 3\hat{i}+2\hat{j}-4\hat{k}$

Let $\vec{r} = (1-2)\hat{i}+(-1+1)\hat{j}+(2-3)\hat{k}$

$\vec{r} = -\hat{i}+0\hat{j}-\hat{k}$

Now $\vec{\tau} = \vec{r}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 0 & -1 \\ 3 & 2 & -4 \end{vmatrix}$

$\vec{\tau} = \hat{i}(0+2) - \hat{j}(4+3) + \hat{k}(-2-0)$

$\vec{\tau} = 2\hat{i}-7\hat{j}-2\hat{k}$ Ans.

Q.12 $\vec{F} = 4\hat{i}-3\hat{k}$

$\vec{r} = \vec{BA} = (2-1)\hat{i}+(-2+3)\hat{j}+(5-1)\hat{k}$

$\vec{r} = \hat{i}+\hat{j}+4\hat{k}$

$\Rightarrow \vec{\tau} = \vec{r}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 4 \\ 4 & 0 & -3 \end{vmatrix}$

$\vec{\tau} = \hat{i}(-3-0)-\hat{j}(-3-16)+\hat{k}(0-4)$

$\vec{\tau} = -3\hat{i}+19\hat{j}-4\hat{k}$ Ans.

Q.13:- $\vec{F} = 2\hat{i}+\hat{j}-3\hat{k}$

$\vec{r} = \vec{BA} = (1-2)\hat{i}+(-2-0)\hat{j}+(1+2)\hat{k}$

$\vec{r} = -\hat{i}-2\hat{j}+3\hat{k}$

$\Rightarrow \vec{\tau} = \vec{r}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -2 & 3 \\ 2 & 1 & -3 \end{vmatrix}$

$\vec{\tau} = (6-3)\hat{i}-\hat{j}(3-6)+\hat{k}(-1+4)$

$\vec{\tau} = 3\hat{i}+3\hat{j}+3\hat{k}$ Ans.

Q.14 Let $\vec{F_1} = \hat{i}-2\hat{j}$, $\vec{F_2} = 3\hat{i}+2\hat{j}-\hat{k}$

$\vec{F_3} = 5\hat{j}+2\hat{k}$

$\vec{F} = \vec{F_1}+\vec{F_2}+\vec{F_3} = 4\hat{i}+5\hat{j}+\hat{k}$

$\vec{r} = \vec{AP} = (2-1)\hat{i}+(0-1)\hat{j}+(1-1)\hat{k}$

$\vec{r} = \hat{i}-\hat{j}+0\hat{k}$

$\Rightarrow \vec{\tau} = \vec{r}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 0 \\ 4 & 5 & 1 \end{vmatrix}$

$\vec{\tau} = (-1-0)\hat{i}-\hat{j}(1-0)+\hat{k}(5+4)$

$\vec{\tau} = -\hat{i}-\hat{j}+9\hat{k}$ Ans.

Q.15 $\vec{F} = 7\hat{i}+4\hat{j}-3\hat{k}$

$\vec{r} = \vec{QP} = (1-2)\hat{i}+(-2-1)\hat{j}+(3-1)\hat{k}$

$\vec{r} = -\hat{i}-3\hat{j}+2\hat{k}$

Thus $\vec{\tau} = \vec{r}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -3 & 2 \\ 7 & 4 & -3 \end{vmatrix}$

$\vec{\tau} = (9-8)\hat{i}-\hat{j}(3-14)+\hat{k}(-4+21)$

$\vec{\tau} = \hat{i}+11\hat{j}+17\hat{k}$ Ans.

The End $\boxed{\text{CH\#7}}$

Tahir Mahmood
M.Sc.(Mathematics)
Govt. College Gujranwala.
Contact No:
0345 6510779