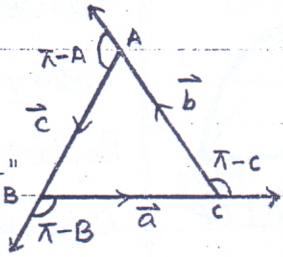
(i) Projection Law by vector method:-

Let $\vec{a}, \vec{b}, \vec{c}$ be the sides of ΔABC.



Using Δ Law of Vector "+"

$$\vec{a}+\vec{b}+\vec{c}=0$$

$$\vec{b}=-\vec{a}-\vec{c}$$

multiplying by $\vec{b}$ scalarly,

$$\vec{b}\cdot\vec{b}=-\vec{a}\cdot\vec{b}-\vec{c}\cdot\vec{b}$$

$$b^2=-ab\cos(\pi-C)-cb\cos(\pi-A)$$

$(\because \cos(\pi-\theta)=-\cos\theta)$

$$b^2=ab\cos C+bc\cos A$$

$\Rightarrow b=a\cos C+c\cos A$ (Proved)

Similarly (ii) Part do yourself.

Exercise 7.4

Q.1 (i) $\vec{a}=2\hat{i}+\hat{j}-\hat{k}$, $\vec{b}=\hat{i}-\hat{j}+\hat{k}$

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 1 & -1\\ 1 & -1 & 1\end{vmatrix}$$

$$\vec{a}\times\vec{b}=\hat{i}(1-1)-\hat{j}(2+1)+\hat{k}(-2-1)$$

$\vec{a}\times\vec{b}=0\hat{i}-3\hat{j}-3\hat{k}$ ______ Ans.

using $\vec{a}\times\vec{b}=-\vec{b}\times\vec{a}$

$$\Rightarrow \vec{b}\times\vec{a}=-(\vec{a}\times\vec{b})$$

$\vec{b}\times\vec{a}=3\hat{j}+3\hat{k}$ ______ Ans.

Now we have to show $\vec{a}, \vec{b}$ are perpendicular to $\vec{a}\times\vec{b}$ and $\vec{b}\times\vec{a}$.

$$\vec{a}\cdot(\vec{a}\times\vec{b})=(2\hat{i}+\hat{j}-\hat{k})\cdot(-3\hat{j}-3\hat{k})$$

$$\vec{a}\cdot(\vec{a}\times\vec{b})=0-3+3=0$$

$\Rightarrow \vec{a}\perp(\vec{a}\times\vec{b})$ ______ Ans.

$$\vec{b}\cdot(\vec{a}\times\vec{b})=(\hat{i}-\hat{j}+\hat{k})\cdot(-3\hat{j}-3\hat{k})$$

$$\vec{b}\cdot(\vec{a}\times\vec{b})=0+3-3=0$$

$\Rightarrow \vec{b}\perp\vec{a}\times\vec{b}$ ______ Ans.

$$\vec{a}\cdot(\vec{b}\times\vec{a})=(2\hat{i}+\hat{j}-\hat{k})\cdot(3\hat{j}+3\hat{k})$$

$$\vec{a}\cdot(\vec{b}\times\vec{a})=0+3-3=0$$

$\Rightarrow \vec{a}\perp\vec{b}\times\vec{a}$ ______ Ans.

$$\vec{b}\cdot(\vec{b}\times\vec{a})=(\hat{i}-\hat{j}+\hat{k})\cdot(3\hat{j}+3\hat{k})$$

$$\vec{b}\cdot(\vec{b}\times\vec{a})=0-3+3=0$$

$\Rightarrow \vec{b}\perp\vec{b}\times\vec{a}$ ______ Ans.

(Similarly Remaining Parts)

Q.2 (i) $\vec{a}=2\hat{i}-6\hat{j}-3\hat{k}$, $\vec{b}=4\hat{i}+3\hat{j}-\hat{k}$

$(\vec{a}\times\vec{b})$ is a vector $\perp$ to $\vec{a}$ & $\vec{b}$ plane

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & -6 & -3\\ 4 & 3 & -1\end{vmatrix}$$

$$(\vec{a}\times\vec{b})=\hat{i}(6+9)-\hat{j}(-2+12)+\hat{k}(6+24)$$

$$\vec{a}\times\vec{b}=15\hat{i}-10\hat{j}+30\hat{k}$$

The unit vector $\perp$ to $\vec{a}$ & $\vec{b}$ plane.

$$(\widehat{\vec{a}\times\vec{b}})=\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$$

$$|\vec{a}\times\vec{b}|=\sqrt{225+100+900}=\sqrt{1225}=35$$

$$(\widehat{\vec{a}\times\vec{b}})=\frac{15\hat{i}-10\hat{j}+30\hat{k}}{35}$$

$$(\widehat{\vec{a}\times\vec{b}})=\frac{3}{7}\hat{i}-\frac{2}{7}\hat{j}+\frac{6}{7}\hat{k}$$

which is unit vector $\perp$ to $\vec{a}, \vec{b}$ plane.

$|\vec{a}| = \sqrt{4+36+9} = \sqrt{49} = 7$

$|\vec{b}| = \sqrt{16+9+1} = \sqrt{26}$

Let θ is the angle between $\vec{a}, \vec{b}$ then

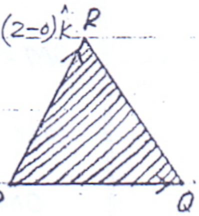
(Similarly Remainings)

$$\sin\theta = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}|\,|\vec{b}|}$$

$$\sin\theta = \frac{35}{(7)(\sqrt{26})} = \frac{5}{\sqrt{26}} \quad \text{Ans.}$$

Q.3 (i) $P(0,0,0)$, $Q(2,3,2)$, $R(-1,1,4)$

Firstly, $\vec{PQ} = (2-0)\hat{i}+(3-0)\hat{j}+(2-0)\hat{k}$



$\vec{PQ} = 2\hat{i}+3\hat{j}+2\hat{k}$

$\vec{PR} = (-1-0)\hat{i}+(1-0)\hat{j}+(4-0)\hat{k}$

$\vec{PR} = -\hat{i}+\hat{j}+4\hat{k}$

Now $\vec{PQ}\times\vec{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 2 \\ -1 & 1 & 4 \end{vmatrix}$

$= \hat{i}(12-2) - \hat{j}(8+2) + \hat{k}(2+3)$

$\vec{PQ}\times\vec{PR} = 10\hat{i} - 10\hat{j} + 5\hat{k}$

$|\vec{PQ}\times\vec{PR}| = \sqrt{100+100+25} = \sqrt{225} = 15$

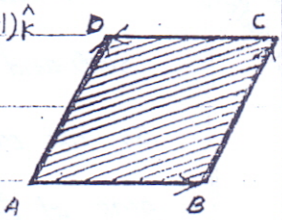
Area of $\Delta PQR = \frac{1}{2}|\vec{PQ}\times\vec{PR}| = \frac{15}{2}$

$\Delta PQR = 15/2$ Sq-units.

{Similarly (ii)}

Q.4 (ii) $A(1,2,-1)$, $B(4,2,-3)$, $C(6,-5,2)$, $D(9,-5,0)$

$\vec{AB} = (4-1)\hat{i}+(2-2)\hat{j}+(-3+1)\hat{k}$



$\vec{AB} = 3\hat{i} - 2\hat{k}$

$\vec{AD} = (9-1)\hat{i}+(-5-2)\hat{j}+(0+1)\hat{k}$

$\vec{AD} = 8\hat{i} - 7\hat{j} + \hat{k}$

Area of ▱ABCD $= |\vec{AB}\times\vec{AD}|$

$\vec{AB}\times\vec{AD} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 0 & -2 \\ 8 & -7 & 1 \end{vmatrix}$

$= \hat{i}(0-14) - \hat{j}(3+16) + \hat{k}(-21-0)$

$\vec{AB}\times\vec{AD} = -14\hat{i} - 19\hat{j} - 21\hat{k}$

$|\vec{AB}\times\vec{AD}| = \sqrt{196+361+441} = \sqrt{998}$

Thus Area of ▱ABCD $= \sqrt{998}$ Sq-Units.

(Similarly Remaining Parts)

Q.5 (i) $\vec{u} = 5\hat{i}-\hat{j}+\hat{k}$, $\vec{v} = \hat{j}-5\hat{k}$, $\vec{w} = -15\hat{i}+3\hat{j}-3\hat{k}$

$\vec{u}\cdot\vec{v} = (5\hat{i}-\hat{j}+\hat{k})\cdot(\hat{j}-5\hat{k}) = -1-5 = -6 \neq 0$

$\vec{u}\cdot\vec{w} = (5\hat{i}-\hat{j}+\hat{k})\cdot(-15\hat{i}+3\hat{j}-3\hat{k}) = -75-3-3 \neq 0$

$\vec{v}\cdot\vec{w} = (\hat{j}-5\hat{k})\cdot(-15\hat{i}+3\hat{j}-3\hat{k}) = 3+15 \neq 0$

$\vec{w} = -3[5\hat{i}-\hat{j}+\hat{k}] = -3\vec{u}$

$\vec{w} \parallel \vec{u}$

Thus no vector is perpendicular to other but $\vec{w}$ is parallel to $\vec{u}$.

(ii) $\vec{u} = \hat{i}+2\hat{j}-\hat{k}$, $\vec{v} = -\hat{i}+\hat{j}+\hat{k}$, $\vec{w} = -\frac{\pi}{2}\hat{i} - \pi\hat{j} + \frac{\pi}{2}\hat{k}$

$\vec{u}\cdot\vec{v} = (\hat{i}+2\hat{j}-\hat{k})\cdot(-\hat{i}+\hat{j}+\hat{k}) = -1+2-1 = 0$
$\Rightarrow \vec{u}\perp\vec{v}$

$u\cdot\vec{w} = (\hat{i}+2\hat{j}-\hat{k})\cdot(-\frac{\pi}{2}\hat{i} - \pi\hat{j} + \frac{\pi}{2}\hat{k}) = -\frac{\pi}{2} - 2\pi - \frac{\pi}{2} \neq 0$

$v\cdot\vec{w} = (-\hat{i}+\hat{j}+\hat{k})\cdot(-\frac{\pi}{2}\hat{i} - \pi j + \frac{\pi}{2}\hat{k}) = \frac{\pi}{2} - \pi + \frac{\pi}{2}$

$= \pi - \pi = 0 \Rightarrow \vec{v}\perp\vec{w}$

$\vec{w} = -\frac{\pi}{2}\{\hat{i}+2\hat{j}-\hat{k}\} = -\frac{\pi}{2}\vec{u}$

$\Rightarrow \vec{w} \parallel \vec{u}$

Thus $\vec{u}\perp\vec{v}$, $\vec{v}\perp\vec{w}$ and $\vec{w}\parallel\vec{u}$.

Q.6 Show that

$\vec{a}\times(\vec{b}+\vec{c})+\vec{b}(\vec{c}+\vec{a})+\vec{c}\times(\vec{a}+\vec{b})=\vec{o}$

$LHS=\vec{a}\times(\vec{b}+\vec{c})+\vec{b}\times(\vec{c}+\vec{a})+\vec{c}\times(\vec{a}+\vec{b})$

$=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}+\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{a}+\vec{c}\times\vec{b}$

using $\vec{u}\times\vec{v}=-\vec{v}\times\vec{u}$ Fact

$=\cancel{\vec{a}\times\vec{b}}+\cancel{\vec{a}\times\vec{c}}+\cancel{\vec{b}\times\vec{c}}-\cancel{\vec{a}\times\vec{b}}-\cancel{\vec{a}\times\vec{c}}-\cancel{\vec{b}\times\vec{c}}$

$=\vec{O}=R.H.S.$

Thus $\vec{a}\times(\vec{b}+\vec{c})+\vec{b}\times(\vec{c}+\vec{a})+\vec{c}\times(\vec{a}+\vec{b})=\vec{o}$

Q.7 $\vec{a}+\vec{b}+\vec{c}=\vec{o}$

$\Rightarrow \vec{a}+\vec{b}=-\vec{c}$

Post multiplying vectorlly by $\vec{b}$

$\vec{a}\times\vec{b}+\vec{b}\times\vec{b}=-\vec{c}\times\vec{b}$

$\vec{a}\times\vec{b}+\vec{o}=\vec{b}\times\vec{c}$ $\left\{\begin{array}{l}\because \vec{a}\times\vec{a}=\vec{o}\\ \because \vec{a}\times\vec{b}=-\vec{b}\times\vec{a}\end{array}\right\}$

$\vec{a}\times\vec{b}=\vec{b}\times\vec{c}$ ——— (i)

Again $\vec{a}+\vec{b}+\vec{c}=\vec{o}$

$\vec{b}+\vec{c}=-\vec{a}$

Post multiplying vectorly by $\vec{c}$

$\vec{b}\times\vec{c}+\vec{c}\times\vec{c}=-\vec{a}\times\vec{c}$

$\vec{b}\times\vec{c}+\vec{o}=\vec{c}\times\vec{a}$

$\vec{b}\times\vec{c}=\vec{c}\times\vec{a}$ ——— (ii)

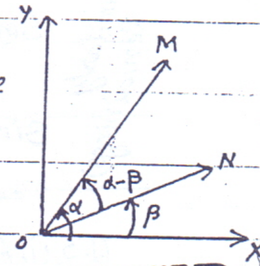
From (i) and (ii)

$\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}$ (Proved)

Q.8 Prove $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$.

Proof:-

Let $\vec{OM}$ and $\vec{ON}$ be the unit vectors in xy plane with α,β angles to x-axis.



$m\angle MON=\alpha-\beta$

$\vec{OM}=\cos\alpha\hat{i}+\sin\alpha\hat{j}$, $\vec{ON}=\cos\beta\hat{i}+\sin\beta\hat{j}$

Also $|\vec{OM}|=|\vec{ON}|=1$.

$\vec{OM}\times\vec{ON}=|\vec{OM}|\,|\vec{ON}|\sin(\alpha-\beta)\hat{n}$

where $\hat{n}=-\hat{k}$ perpendicular to xy plane (inwards.)

$$(1)(1)\sin(\alpha-\beta)(-\hat{k})=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \cos\alpha & \sin\alpha & 0\\ \cos\beta & \sin\beta & 0\end{vmatrix}$$

$\sin(\alpha-\beta)(-\hat{k})=\hat{k}\{\cos\alpha\sin\beta-\sin\alpha\cos\beta\}$

$(-\hat{k})\sin(\alpha-\beta)=(-\hat{k})(\sin\alpha\cos\beta-\cos\alpha\sin\beta)$

$\Rightarrow \sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$ (Proved)

Q.9 $\vec{a}\times\vec{b}=|\vec{a}|\,|\vec{b}|\sin\theta\,\hat{n}=\vec{o}$ ——— (i)

$\vec{a}\cdot\vec{b}=|\vec{a}|\,|\vec{b}|\cos\theta=0$ ——— (ii)

From (i) and (ii), It is obvious that $\sin\theta$ and $\cos\theta$ cannot be zero together so we can conclude that atleast one of $\vec{a},\vec{b}$ is zero vector or may both zero vectors.
