TAHIR MEHMOOD
M.Sc Math
0345-6510779

$$x^2\left(\frac{2}{10}+\frac{18}{10}+\frac{90}{10}\right)+y^2\left(\frac{18}{10}+\frac{10}{10}-\frac{18}{10}\right)-11=0$$

$$x^2\left(\frac{110}{10}\right)+y^2\left(\frac{10}{10}\right)-11=0$$

$$\boxed{11x^2+y^2-11=0}$$

(ii) $xy+4x-3y-10=0$

$\because x = X\cos\theta - Y\sin\theta,\ y = X\sin\theta + Y\cos\theta$

So

$(X\cos\theta - Y\sin\theta)(X\sin\theta+Y\cos\theta)+4(X\cos\theta-Y\sin\theta)$
$-3(X\sin\theta+Y\cos\theta)-10=0$

$\Rightarrow X^2\cos\theta\sin\theta - Y^2\sin\theta\cos\theta - XY\sin^2\theta + XY\cos^2\theta$
$+4X\cos\theta - 4Y\sin\theta - 3X\sin\theta - 3Y\cos\theta - 10 = 0$

Let $\cos^2\theta - \sin^2\theta = 0$

$\Rightarrow \cos^2\theta = \sin^2\theta \Rightarrow \tan^2\theta = 1$

$\tan\theta = 1 \Rightarrow \theta = \tan^{-1}(1) = 45^\circ$

So

$\sin\theta = \frac{1}{\sqrt2}$ and $\cos\theta = \frac{1}{\sqrt2}$

$\Rightarrow X^2\left(\frac{1}{\sqrt2}\right)\left(\frac{1}{\sqrt2}\right) - Y^2\left(\frac{1}{\sqrt2}\right)\left(\frac{1}{\sqrt2}\right)+4X\left(\frac{1}{\sqrt2}\right)$
$-4Y\left(\frac{1}{\sqrt2}\right)-3X\left(\frac{1}{\sqrt2}\right)-3Y\left(\frac{1}{\sqrt2}\right)-10=0$

$\Rightarrow \frac{X^2}{2}-\frac{Y^2}{2}+\frac{4}{\sqrt2}X-\frac{4}{\sqrt2}Y-\frac{3X}{\sqrt2}-\frac{3Y}{\sqrt2}-10=0$

Multiplying by 2, we have

$X^2-Y^2+4\sqrt2X-4\sqrt2Y-3\sqrt2X-3\sqrt2Y-20=0$

$$\boxed{X^2-Y^2+\sqrt2X-7\sqrt2Y-20=0}$$

(iii) $5x^2-6xy+5y^2-8=0$

$\because x = X\cos\theta - Y\sin\theta,\ y = X\sin\theta + Y\cos\theta$

So transformed Equation is:

$5[X\cos\theta-Y\sin\theta]^2-6[X\cos\theta-Y\sin\theta][X\sin\theta+Y\cos\theta]$
$+5[X\sin\theta+Y\cos\theta]^2-8=0$

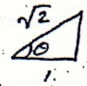
$\Rightarrow 5[X^2\cos^2\theta+Y^2\sin^2\theta-2XY\sin\theta\cos\theta]$
$-6[X^2\sin\theta\cos\theta-XY\sin^2\theta+XY\cos^2\theta-Y^2\sin\theta\cos\theta]$
$+5[X^2\sin^2\theta+Y^2\cos^2\theta+2XY\sin\theta\cos\theta]-8=0$

Putting the Coeff. of XY zero.

$-10\sin\theta\cos\theta+\cos^2\theta-\sin^2\theta+10\sin\theta\cos\theta=0$

$\Rightarrow \cos^2\theta-\sin^2\theta=0 \Rightarrow \tan^2\theta=1$

$\Rightarrow \tan\theta = 1$



So

$\sin\theta=\frac{1}{\sqrt2}\quad \cos\theta=\frac{1}{\sqrt2}$

$\Rightarrow 5\left[X^2\left(\frac12\right)+Y^2\left(\frac12\right)\right]-6\left[X^2\left(\frac{1}{\sqrt2}\right)\left(\frac{1}{\sqrt2}\right)\right.$
$\left.-Y^2\left(\frac{1}{\sqrt2}\right)\left(\frac{1}{\sqrt2}\right)\right]+5\left[X^2\left(\frac12\right)+Y^2\left(\frac12\right)\right]-8=0$

$\Rightarrow \frac52X^2+\frac52Y^2-\frac62X^2+\frac62Y^2+\frac{5X^2}{2}+\frac{5Y^2}{2}-8=0$

Multiplying by 2, we have

$5X^2+5Y^2-6X^2+6Y^2+5X^2+5Y^2-16=0$

$4X^2+16Y^2-16=0$

$\boxed{X^2+4Y^2-4=0}\quad \because 4\neq0$

General Equation of Conics :- (With Rotation)

The second degree equation

$$ax^2+by^2+2hxy+2gx+2fy+c=0 \quad \text{—(1)}$$

is called General Equation of Conics.

In this Equation

h^2-ab is called discriminant of Conics Equation.

If

(i) $h^2-ab<0$ then Conics is circle or ellipse.

(ii) $h^2-ab=0$ then Conics is parabola.

(iii) $h^2-ab>0$ then Conics is hyperbola.

This equation can be transformed into rotated plane as:

$$AX^2+BY^2+2GX+2FY+C=0$$

with $2\theta=\tan^{-1}\left(\frac{2h}{a-b}\right)$

for $0^\circ<\theta\leq 90^\circ$

* If $a=b$ or $a=0=b$ then $\theta=45^\circ$

If $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=0$ then Equation (1) represents a pair of Straight Lines.

TAHIR

TAHIR MEHMOOD
M.Sc Math
0345-6510779

Equation of tangent to Conic Equation

Show that equation of tangent to $ax^2+by^2+2hxy+2gx+2fy+c=0$ at (x_1,y_1) is $axx_1+byy_1+h(xy_1+x_1y)+g(x+x_1)+f(y+y_1)+c=0$

Proof:- Consider $ax^2+by^2+2hxy+2gx+2fy+c=0$

Diff w.r.t. "x", we have

$$2ax+2by\frac{dy}{dx}+2hy+2hx\frac{dy}{dx}+2g+2f\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=-\frac{ax+hy+g}{hx+by+f} \Rightarrow \left.\frac{dy}{dx}\right]_{(x_1,y_1)}=-\frac{ax_1+hy_1+g}{hx_1+by_1+f}$$

which is slope of tangent.

Equation of tangent through (x_1,y_1) is:

$$y-y_1=\left.\frac{dy}{dx}\right]_{(x_1,y_1)}(x-x_1) \Rightarrow y-y_1=-\frac{(ax_1+hy_1+g)}{(hx_1+by_1+f)}(x-x_1)$$

$$\Rightarrow (y-y_1)(hx_1+by_1+f)=-(ax_1+hy_1+g)(x-x_1)$$

$$\Rightarrow hx_1y-hx_1y_1+byy_1-by_1^2+yf-y_1f=ax_1^2-axx_1-hxy_1+hx_1y_1-gx+gx_1$$

$$\Rightarrow axx_1+byy_1+hxy_1+hx_1y+gx+fy=ax_1^2+by_1^2+2hx_1y_1+gx_1+fy_1$$

$$\Rightarrow axx_1+byy_1+h(xy_1+x_1y)+gx+fy=ax_1^2+by_1^2+2hx_1y_1+gx_1+fy_1$$

Adding gx_1+fy_1+c in both sides.

$$\Rightarrow axx_1+byy_1+h(xy_1+x_1y)+g(x+x_1)+f(y+y_1)+c=ax_1^2+by_1^2+2hx_1y_1+2gx_1+2fy_1+c$$

$\because (x_1,y_1)$ lies on Eq so $ax_1^2+by_1^2+2hx_1y_1+2gx_1+2fy_1+c=0$

$$\Rightarrow axx_1+byy_1+h(xy_1+x_1y)+g(x+x_1)+f(y+y_1)+c=0$$

which is required Equation of tangent.

The general Equation of Conic without rotation

$$Ax^2+By^2+Gx+Fy+C=0 \text{ is}$$

(i) Circle if $A=B\neq 0$

(ii) Ellipse if $A\neq B$ but have same Signs

(iii) Hyperbola if $A\neq B$ but have opposite Signs.

(iv) Parabola if $A=0$ or $B=0$ but not both simultaneously.

Equation of tangent can be obtained as:

$x^2\Rightarrow xx_1$, $y^2\Rightarrow yy_1$

$2x\Rightarrow x+x_1$, $2y\Rightarrow y+y_1$

$2xy\Rightarrow xy_1+yx_1$ replacements in the given Equation at $P(x_1,y_1)$.

TAHIR

TAHIR MEHMOOD
M.Sc Math
0345-6510779

Exercise : 6.9

Q.2 Show that
(i) $10xy+8x-15y-12=0$
(ii) $6x^2+xy-y^2-21x-8y+9=0$
represents a pair of straight lines. Find equations of these lines.

Sol:- (i) $10xy+8x-15y-12=0$

Comparing with standard equation

$a=0,\ b=0,\ h=5,\ g=4,\ f=\frac{-15}{2},\ c=-12$

Consider

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 0 & 5 & 4 \\ 5 & 0 & -15/2 \\ 4 & -15/2 & -12 \end{vmatrix}$$

$$= 0-5\begin{vmatrix} 5 & -15/2 \\ 4 & -12 \end{vmatrix} + 4\begin{vmatrix} 5 & 0 \\ 4 & -15/2 \end{vmatrix}$$

$$= 0-5(-60+30)+4\left(\frac{-75}{2}-0\right)$$

$$= 0-5(-30)-75\times 2$$

$$= 0+150-150=0$$

So given conic is pair of lines.

Now $10xy-15y+8x-12=0$

$$5y(2x-3)+4(2x-3)=0$$

$$(2x-3)(5y+4)=0$$

So required lines are

$$2x-3=0 \text{ and } 5y+4=0$$

(ii) $6x^2-y^2+xy-21x-8y+9=0$

Comparing with standard equation.

$a=6,\ b=-1,\ h=\frac{1}{2},\ g=\frac{-21}{2},\ f=-4,\ c=9$

Consider

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 6 & 1/2 & -21/2 \\ 1/2 & -1 & -4 \\ \frac{-21}{2} & -4 & 9 \end{vmatrix}$$

$$= 6\begin{vmatrix} -1 & -4 \\ -4 & 9 \end{vmatrix} - \frac{1}{2}\begin{vmatrix} \frac{1}{2} & -4 \\ \frac{-21}{2} & 9 \end{vmatrix} - \frac{21}{2}\begin{vmatrix} \frac{1}{2} & -1 \\ \frac{-21}{2} & -4 \end{vmatrix}$$

$$= 6(-9-16)-\frac{1}{2}\left(\frac{9}{2}-42\right)-\frac{21}{2}\left(-2-\frac{21}{2}\right)$$

$$= -150-\frac{1}{2}\left(\frac{-75}{2}\right)-\frac{21}{2}\left(\frac{-25}{2}\right)$$

$$= -150+\frac{75}{4}+\frac{525}{4} = -150+\frac{600}{4}$$

$$= -150+150=0$$

So given conic is pair of straight lines.

Now $6x^2+(y-21)x+(9-8y-y^2)=0$

which is quadratic in x

$$x=\frac{-(y-21)\pm\sqrt{(y-21)^2-4(6)(9-8y-y^2)}}{2(6)}$$

$$x=\frac{-y+21\pm\sqrt{y^2+441-42y-216+192y+24y^2}}{12}$$

$$x=\frac{-y+21\pm\sqrt{25y^2+150y+225}}{12}$$

$$x=\frac{-y+21\pm\sqrt{(5y+15)^2}}{12}$$

$$x=\frac{-y+21\pm(5y+15)}{12}$$

$$\Rightarrow x=\frac{-y+21+5y+15}{12} \text{ and } x=\frac{-y+21-5y-15}{12}$$

$$x=\frac{4y+36}{12} \text{ and } x=\frac{-6y+6}{12}$$

$$x=\frac{y+9}{3} \text{ and } x=\frac{-y+1}{2}$$

$$3x=y+9 \text{ and } 2x=-y+1$$

So $3x-y-9=0$ and $2x+y-1=0$

Q.3 Find equation of tangent at indicated pt.

(i) $3x^2-7y^2+2x-y-48=0$ at $(4,1)$.

Sol:- Diff. w.r.t. "x"

$$6x-7\left(2y\frac{dy}{dx}\right)+2-\frac{dy}{dx}=0$$

$$6x+2-(14y+1)\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=\frac{6x+2}{14y+1}$$

$$\left.\frac{dy}{dx}\right]_{(4,1)}=\frac{6(4)+2}{14(1)+1}=\frac{26}{15}$$

Eq of tangent through $(4,1)$:

$$y-1=\frac{26}{15}(x-4) \Rightarrow 15(y-1)=26(x-4)$$

$$\Rightarrow 15y-15=26x-104$$

$$\Rightarrow 26x-15y-104+15=0$$

$$\Rightarrow \boxed{26x-15y-89=0}$$ which is required.

Msc (math) TAHIR

TAHIR MEHMOOD
M.Sc Math
0345-6510779

(ii) $x^2+5xy-4y^2+4=0$ at $y=-1$

Putting in equation, we have

$x^2-5x-4+4=0$

$\Rightarrow x^2-5x=0 \Rightarrow x(x-5)=0$

$\Rightarrow x=0$ and $x=5$

So points are $(0,-1)$ and $(5,-1)$

Diff. w.r.t. "x"

$2x+5y+5x\frac{dy}{dx}-8y\frac{dy}{dx}=0$

$2x+5y=(8y-5x)\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{2x+5y}{8y-5x}$

$\left.\frac{dy}{dx}\right|_{(0,-1)} = \frac{2(0)+5(-1)}{8(-1)-5(0)} = \frac{-5}{-8} = 5/8$

$\left.\frac{dy}{dx}\right|_{(5,-1)} = \frac{2(5)+5(-1)}{8(-1)-5(5)} = \frac{10-5}{-8-25} = \frac{-5}{33}$

Equation of tangent through $(0,-1)$

$y+1=\frac{5}{8}(x-0) \Rightarrow 8(y+1)=5x$

$\Rightarrow \boxed{5x-8y-8=0}$ Ans.

Eq. of tangent through $(5,-1)$

$y+1=\frac{-5}{33}(x-5)$

$\Rightarrow 33(y+1)=-5x+25$

$\Rightarrow 5x+33y+33-25=0$

$\Rightarrow \boxed{5x+33y+8=0}$ Ans.

(iii) $x^2+4xy-3y^2-5x-9y+6=0$ at $x=3$

Putting in Equation, we have

$9+12y-3y^2-15-9y+6=0$

$-3y^2+3y=0 \Rightarrow -3y(y-1)=0$

$\because -3\neq 0 \Rightarrow y=0$ and $y=1$

So points are $(3,0)$ and $(3,1)$

Diff w.r.t. "x", we have

$2x+4y+4x\frac{dy}{dx}-6y\frac{dy}{dx}-5-9\frac{dy}{dx}=0$

$\frac{dy}{dx}=\frac{5-2x-4y}{4x-6y-9}$

$\left.\frac{dy}{dx}\right]_{(3,0)} = \frac{5-2(3)-0}{4(3)-0-9} = \frac{-1}{3}$

$\left.\frac{dy}{dx}\right]_{(3,1)} = \frac{5-2(3)-4}{4(3)-6-9} = \frac{-5}{-3} = 5/3$

Eq of tangent through $(3,0)$

$y-0=\frac{-1}{3}(x-3)$

$\Rightarrow 3y=-x+3$

$\Rightarrow \boxed{x+3y-3=0}$ Ans.

Equation of tangent through $(3,1)$

$y-1=\frac{5}{3}(x-3)$

$\Rightarrow 3y-3=5x-15$

$\Rightarrow 5x-3y-15+3=0$

$\Rightarrow \boxed{5x-3y-12=0}$ Ans.

Q.1 is similar to Last Question of Exercise 6.8
Student should try to solve it.