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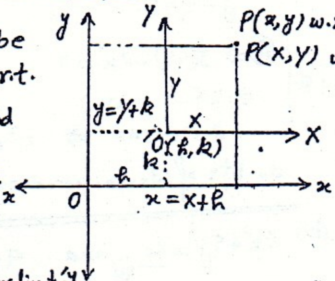
Translation of Axes:-

Let $P(x,y)$ be any point w.r.t. xy-plane and Suppose Axes are translated through $O'(h,k)$ then the Co-ordinates w.r.t. new origion $O'(h,k)$ are

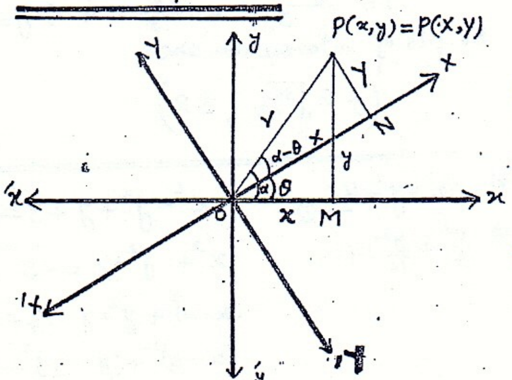


$\boxed{X = x-h}$ and $\boxed{Y = y-k}$

Now Co-ordinates of old axis subject to new axis are

$\boxed{x = X+h}$ and $\boxed{y = Y+k}$

Rotation of Axes:-



Let $P(x,y)$ be any point w.r.t. xy plane Such that $x = r\cos\alpha$ and $y = r\sin\alpha$

Suppose Axes are rotated at an angle θ where $0 < \theta \le \pi/2$.

Let $P(X,Y)$ be the Co-ordinates w.r.t. new axes then

$X = r\cos(\alpha-\theta)$

$X = (r\cos\alpha)\cos\theta + (r\sin\alpha)\sin\theta$

$\boxed{X = x\cos\theta + y\sin\theta}$

and $Y = r\sin(\alpha-\theta)$

$Y = (r\sin\alpha)\cos\theta - (r\cos\alpha)\sin\theta$

$\boxed{Y = y\cos\theta - x\sin\theta}$

Now we are to find the co-ordines of P w.r.t. old axis when the Co-ordinates of new axes are given then

$X\sin\theta = x\sin\theta\cancel{\cos\theta} + y\sin^2\theta$

$Y\cos\theta = \cancel{-x\sin\theta\cos\theta} + y\cos^2\theta$

$X\sin\theta + Y\cos\theta = y(\sin^2\theta + \cos^2\theta)$

$\Rightarrow \boxed{y = X\sin\theta + Y\cos\theta}$

Now

$X\cos\theta = x\cos^2\theta + \cancel{y\sin\theta\cos\theta}$

$-Y\sin\theta = \mp x\sin^2\theta \pm \cancel{y\sin\theta\cos\theta}$

$X\cos\theta - Y\sin\theta = x(\cos^2\theta + \sin^2\theta)$

$\Rightarrow \boxed{x = X\cos\theta - Y\sin\theta}$

Exercise: 6.8

Q.1 Find the Equation translated to:

(i) $x^2+16y-16=0$ $\quad O'(h,k)=O'(0,1)$

Now $X = x-0 = x$ and $Y = y-1$

$\Rightarrow x = X$ and $y = Y+1$

So Eq w.r.t. XY-plane is

$X^2+16(Y+1)-16=0$

$\Rightarrow X^2+16Y+16-16=0$

$\Rightarrow \boxed{X^2+16Y=0}$

(ii) $4x^2+y^2+16x-10y+37=0$ $\quad O'(h,k)=O'(-2,5)$

Now $X = x+2$ and $Y = y-5$

$x = X-2$ and $y = Y+5$

So Eq w.r.t. XY-plane is:

$4(X-2)^2+(Y+5)^2+16(X-2)-10(Y+5)+37=0$

$\Rightarrow 4(X^2+4-4X)+(Y^2+25+10Y)+16X-32-10Y-50+37=0$

$\Rightarrow 4X^2+Y^2-16X+16X+10Y-10Y+16+25-32+37-50=0$

$\Rightarrow \boxed{4X^2+Y^2-4=0}$

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(iii) $9x^2+4y^2+18x-16y-11=0 \quad O'(-1,2)$

Now $X=x+1$ and $Y=y-2$

$\Rightarrow x=X-1$ and $y=Y+2$

Now eq w.r.t. XY-plane is:

$9(X-1)^2+4(Y+2)^2+18(X-1)-16(Y+2)-11=0$

$9(X^2-2X+1)+4(Y^2+4Y+4)+18X-18-16Y-32-11=0$

$9X^2-18X+9+4Y^2+16Y+16+18X-18-16Y-32-11=0$

$\Rightarrow \boxed{9X^2+4Y^2-36=0}$

(iv) $x^2-y^2+4x+8y-11=0 \quad O'(-2,4)$

Now $x=X-2$ and $y=Y+4$

Now Eq w.r.t. XY-plane is:

$(X-2)^2-(Y+4)^2+4(X-2)+8(Y+4)-11=0$

$X^2+4-4X-Y^2-16-8Y+4X-8+8Y+32-11=0$

$\Rightarrow \boxed{X^2-Y^2+1=0}$

(v) $9x^2-4y^2+36x+8y-4=0 \quad O'(-2,1)$

Now $x=X-2$ and $y=Y+1$

Now Eq w.r.t. XY plane is:

$9(X-2)^2-4(Y+1)^2+36(X-2)+8(Y+1)-4=0$

$9(X^2-4X+4)-4(Y^2+2Y+1)+36X-72+8Y+8-4=0$

$9X^2-36X+36-4Y^2-8Y-4+36X-72+8Y+8-4=0$

$\boxed{9X^2-4Y^2-36=0}$

Q.2:- *Find the Co-ordinates of new origin for which first degree terms are removed. Also find transformed Equation.*

(i) $3x^2-2y^2+24x+12y+24=0$

Consider $x=X+h$ and $y=Y+k$

so transformed Equation is:

$3(X+h)^2-2(Y+k)^2+24(X+h)+12(Y+k)+24=0$

$3(X^2+h^2+2Xh)-2(Y^2+k^2+2Yk)+24X+24h+12Y+12k+24=0$

$3X^2-2Y^2+(6h+24)X+(12-4k)Y+3h^2-2k^2+24h+12k+24=0$

Consider $6h+24=0 \quad 12-4k=0$

$\Rightarrow h=-4 \quad k=3$

So $3X^2-2Y^2+48-18-96+36+24=0$

$3X^2-2Y^2-6=0$ with $O'(-4,3)$

(ii) $25x^2+9y^2+50x-36y-164=0$

Consider $x=X+h$ and $y=Y+k$

So transformed Equation is:

$25(X+h)^2+9(Y+k)^2+50(X+h)-36(Y+k)-164=0$

$25(X^2+h^2+2Xh)+9(Y^2+k^2+2Yk)+50X+50h-36Y-36k-164=0$

$\Rightarrow 25X^2+9Y^2+(50h+50)X+(18k-36)k+25h^2+9k^2+50h-36k-164=0$

Consider $50h+50=0$ and $18k-36=0$

$\Rightarrow h=-1$ and $k=2$

So $25X^2+9Y^2+25+36-50-72-164=0$

$25X^2+9Y^2-225=0$ with $O'(-1,2)$

(iii) $x^2-y^2-6x+2y+7=0$

Consider $x=X+h$ and $y=Y+k$

so transformed Equation is:

$(X+h)^2-(Y+k)^2-6(X+h)+2(Y+k)+7=0$

$(X^2+h^2+2Xh)-(Y^2+k^2+2Yk)-6X-6h+2Y+2k+7=0$

$X^2-Y^2+(2h-6)X+(2-2k)Y+h^2-k^2-6h+2k+7=0$

Consider $2h-6=0$ and $2-2k=0$

$\Rightarrow h=3$ and $k=1$

So $X^2-Y^2+9-1-18+2+7=0$

$X^2-Y^2-1=0$ with $O'(3,1)$

Q.3 *Find an Equation about new axes:*

(i) $xy=1 \quad \theta=45^\circ$

$\because \cos\theta=\cos 45^\circ=\frac{1}{\sqrt{2}}$ and $\sin\theta=\sin 45^\circ=\frac{1}{\sqrt{2}}$

Now $x=X\cos\theta-Y\sin\theta=\frac{X}{\sqrt{2}}-\frac{Y}{\sqrt{2}}=\frac{X-Y}{\sqrt{2}}$

$y=X\sin\theta+Y\cos\theta=\frac{X}{\sqrt{2}}+\frac{Y}{\sqrt{2}}=\frac{X+Y}{\sqrt{2}}$

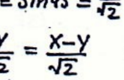
Now Equation about rotated axes:

$\left(\frac{X-Y}{\sqrt{2}}\right)\left(\frac{X+Y}{\sqrt{2}}\right)=1 \Rightarrow \frac{X^2-Y^2}{2}=1$

$\Rightarrow \boxed{X^2-Y^2=2}$

(ii) $7x^2-8xy+y^2-9=0 \quad \theta=\tan^{-1}2$

$\Rightarrow \tan\theta=2$



$\sin\theta=\frac{2}{\sqrt{5}} \quad \cos\theta=\frac{1}{\sqrt{5}}$

$x=X\cos\theta-Y\sin\theta=\frac{X}{\sqrt{5}}-\frac{2Y}{\sqrt{5}}=\frac{X-2Y}{\sqrt{5}}$

$y=X\sin\theta+Y\cos\theta=\frac{2X}{\sqrt{5}}+\frac{Y}{\sqrt{5}}=\frac{2X+Y}{\sqrt{5}}$

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$$7\left[\frac{X-2Y}{\sqrt5}\right]^2-8\left[\frac{X-2Y}{\sqrt5}\right]\left[\frac{2X+Y}{\sqrt5}\right]+\left[\frac{2X+Y}{\sqrt5}\right]^2-9=0$$

$$\Rightarrow \frac{7}{5}\left[X^2+4Y^2-4XY\right]-\frac{8}{5}\left[2X^2+XY-4XY-2Y^2\right]+\frac{4X^2+Y^2+4XY}{5}-9=0$$

Multiplying both sides by 5

$$7X^2+28Y^2-28XY-16X^2+24XY+16Y^2+4X^2+Y^2+4XY-45=0$$

$$-5X^2+45Y^2-45=0$$

Dividing by -5, we have

$$\boxed{X^2-9Y^2+9=0}$$

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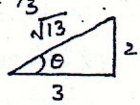
(iii) $9x^2+12xy+4y^2-x-y=0$, $\theta=\tan^{-1}2/3$

$\because \theta=\tan^{-1}(2/3) \Rightarrow \tan\theta=2/3$

$\sin\theta=\frac{2}{\sqrt{13}}$ $\quad\cos\theta=\frac{3}{\sqrt{13}}$



$$x=X\cos\theta-Y\sin\theta=\frac{3X}{\sqrt{13}}-\frac{2Y}{\sqrt{13}}=\frac{3X-2Y}{\sqrt{13}}$$

$$y=X\sin\theta+Y\cos\theta=\frac{2X}{\sqrt{13}}+\frac{3Y}{\sqrt{13}}=\frac{2X+3Y}{\sqrt{13}}$$

$$\Rightarrow 9\left(\frac{3X-2Y}{\sqrt{13}}\right)^2+12\left(\frac{3X-2Y}{\sqrt{13}}\right)\left(\frac{2X+3Y}{\sqrt{13}}\right)+4\left(\frac{2X+3Y}{\sqrt{13}}\right)^2-\frac{3X-2Y}{\sqrt{13}}-\frac{2X+3Y}{\sqrt{13}}=0$$

$$\Rightarrow \frac{9}{13}\left[9X^2+4Y^2-12XY\right]+\frac{12}{13}\left[6X^2+9XY-4XY-6Y^2\right]+\frac{4}{13}\left[4X^2+9Y^2+12XY\right]-\frac{3X}{\sqrt{13}}+\frac{2Y}{\sqrt{13}}-\frac{2X}{\sqrt{13}}-\frac{3Y}{\sqrt{13}}=0$$

$\Rightarrow$ Multiplying by 13, we have

$$81X^2+36Y^2-108XY+72X^2+60XY-72Y^2+16X^2+36Y^2+48XY-3\sqrt{13}X+2\sqrt{13}Y-2\sqrt{13}X-3\sqrt{13}Y=0$$

$$\Rightarrow 169X^2-5\sqrt{13}X-\sqrt{13}Y=0$$

Divinding by $\sqrt{13}$, we have

$$\boxed{13\sqrt{13}X^2-5X-Y=0}$$

(iv) $x^2-2xy+y^2-2\sqrt2x-2\sqrt2y+2=0$

$\theta=45^\circ$

$\Rightarrow \sin\theta=\frac{1}{\sqrt2}$ and $\cos\theta=\frac{1}{\sqrt2}$

$$y=X\sin\theta-Y\cos\theta=\frac{X}{\sqrt2}-\frac{Y}{\sqrt2}=\frac{X-Y}{\sqrt2}$$

$$x=X\cos\theta+Y\sin\theta=\frac{X}{\sqrt2}+\frac{Y}{\sqrt2}=\frac{X+Y}{\sqrt2}$$

So

$$\left[\frac{X+Y}{\sqrt2}\right]^2-2\left[\frac{X+Y}{\sqrt2}\right]\left[\frac{X-Y}{\sqrt2}\right]+\left[\frac{X-Y}{\sqrt2}\right]^2-2\sqrt2\left[\frac{X+Y}{\sqrt2}\right]-2\sqrt2\left[\frac{X-Y}{\sqrt2}\right]+2=0$$

$$\Rightarrow \frac{X^2+Y^2+2XY}{2}-\frac{2(X^2-Y^2)}{2}+\frac{X^2+Y^2-2XY}{2}-2[X+Y]-2[X-Y]+2=0$$

Multiplying throughout by 2

$$X^2+Y^2+2XY-2X^2+2Y^2+X^2+Y^2-2XY-4X-4Y-4X+4Y+4=0$$

$$4Y^2-8X+4=0$$

$$2Y^2-4X+2=0$$

$$\boxed{Y^2-2X+1=0}$$

Q.4 *Find angle of rotation and also find transformed equation if XY terms are removed.*

(i) $2x^2+6xy+10y^2-11=0$

$\because x=X\cos\theta-Y\sin\theta$ and $y=X\sin\theta+Y\cos\theta$

So transformed Equation is:

$$2[X\cos\theta-Y\sin\theta]^2+6[X\cos\theta-Y\sin\theta][X\sin\theta+Y\cos\theta]+10[X\sin\theta+Y\cos\theta]^2-11=0$$

$$\Rightarrow 2[X^2\cos^2\theta+Y^2\sin^2\theta-2XY\sin\theta\cos\theta]+6[X^2\sin\theta\cos\theta+XY\cos^2\theta-XY\sin^2\theta-Y^2\sin\theta\cos\theta]+10[X^2\sin^2\theta+Y^2\cos^2\theta+2XY\sin\theta\cos\theta]-11=0$$

$$\Rightarrow X^2[2\cos^2\theta+6\sin\theta\cos\theta+10\sin^2\theta]+Y^2[2\sin^2\theta+10\cos^2\theta-6\sin\theta\cos\theta]+XY[16\sin\theta\cos\theta+6\cos^2\theta-6\sin^2\theta]-11=0 \quad (1)$$

Let $16\sin\theta\cos\theta+6(\cos^2\theta-\sin^2\theta)=0$

Dividing by $-2\cos^2\theta$

$$3\tan^2\theta-8\tan\theta-3=0$$

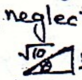
$$3\tan^2\theta-9\tan\theta+\tan\theta-3=0$$

$$3\tan\theta(\tan\theta-3)+1(\tan\theta-3)=0$$

$$(\tan\theta-3)(3\tan\theta+1)=0$$

$\Rightarrow \tan\theta=3$ $\qquad \tan\theta=-1/3 \notin (0,\pi/2)$ So neglecting it

$\sin\theta=\frac{3}{\sqrt{10}}$ $\quad \cos\theta=\frac{1}{\sqrt{10}}$



Putting in (1)

$$\Rightarrow X^2\left[2\left(\tfrac{1}{10}\right)+6\left(\tfrac{1}{\sqrt{10}}\right)\left(\tfrac{3}{\sqrt{10}}\right)+10\left(\tfrac{9}{10}\right)\right]+Y^2\left[2\left(\tfrac{9}{10}\right)+10\left(\tfrac{1}{10}\right)-6\left(\tfrac{3}{\sqrt{10}}\right)\left(\tfrac{1}{\sqrt{10}}\right)\right]-11=0$$

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$$x^2\left(\frac{2}{10}+\frac{18}{10}+\frac{90}{10}\right)+y^2\left(\frac{18}{10}+\frac{10}{10}-\frac{18}{10}\right)-11=0$$

$$X^2\left(\frac{110}{10}\right)+Y^2\left(\frac{10}{10}\right)-11=0$$

$$\boxed{11X^2+Y^2-11=0}$$

(ii) $xy+4x-3y-10=0$

$\because x = X\cos\theta - Y\sin\theta,\ y = X\sin\theta + Y\cos\theta$

So

$$(X\cos\theta - Y\sin\theta)(X\sin\theta+Y\cos\theta)+4(X\cos\theta-Y\sin\theta)-3(X\sin\theta+Y\cos\theta)-10=0$$

$$\Rightarrow X^2\cos\theta\sin\theta - Y^2\sin\theta\cos\theta - XY\sin^2\theta + XY\cos^2\theta + 4X\cos\theta - 4Y\sin\theta - 3X\sin\theta - 3Y\cos\theta - 10 = 0$$

Let $\cos^2\theta - \sin^2\theta = 0$

$\Rightarrow \cos^2\theta = \sin^2\theta \Rightarrow \tan^2\theta = 1$

$\tan\theta = 1 \Rightarrow \theta = \tan^{-1}(1) = 45^\circ$

So $\sin\theta = \frac{1}{\sqrt2}$ and $\cos\theta = \frac{1}{\sqrt2}$

$$\Rightarrow X^2\left(\tfrac{1}{\sqrt2}\right)\left(\tfrac{1}{\sqrt2}\right)-Y^2\left(\tfrac{1}{\sqrt2}\right)\left(\tfrac{1}{\sqrt2}\right)+4X\left(\tfrac{1}{\sqrt2}\right)-4Y\left(\tfrac{1}{\sqrt2}\right)-3X\left(\tfrac{1}{\sqrt2}\right)-3Y\left(\tfrac{1}{\sqrt2}\right)-10=0$$

$$\Rightarrow \frac{X^2}{2}-\frac{Y^2}{2}+\frac{4}{\sqrt2}X-\frac{4}{\sqrt2}Y-\frac{3X}{\sqrt2}-\frac{3Y}{\sqrt2}-10=0$$

Multiplying by 2, we have

$$X^2-Y^2+4\sqrt2X-4\sqrt2Y-3\sqrt2X-3\sqrt2Y-20=0$$

$$\boxed{X^2-Y^2+\sqrt2X-7\sqrt2Y-20=0}$$

(iii) $5x^2-6xy+5y^2-8=0$

$\because x = X\cos\theta - Y\sin\theta,\ y = X\sin\theta + Y\cos\theta$

So transformed Equation is:

$$5[X\cos\theta-Y\sin\theta]^2-6[X\cos\theta-Y\sin\theta][X\sin\theta+Y\cos\theta]+5[X\sin\theta+Y\cos\theta]^2-8=0$$

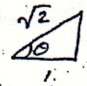
$$\Rightarrow 5[X^2\cos^2\theta+Y^2\sin^2\theta-2XY\sin\theta\cos\theta]-6[X^2\sin\theta\cos\theta-XY\sin^2\theta+XY\cos^2\theta-Y^2\sin\theta\cos\theta]+5[X^2\sin^2\theta+Y^2\cos^2\theta+2XY\sin\theta\cos\theta]-8=0$$

Putting the Coeff. of XY zero.

$$-10\sin\theta\cos\theta+\cos^2\theta-\sin^2\theta+10\sin\theta\cos\theta=0$$

$$\Rightarrow \cos^2\theta-\sin^2\theta=0 \Rightarrow \tan^2\theta=1$$

$\Rightarrow \tan\theta = 1$



So $\sin\theta = \frac{1}{\sqrt2}$ $\cos\theta = \frac{1}{\sqrt2}$

$$\Rightarrow 5\left[X^2\left(\tfrac12\right)+Y^2\left(\tfrac12\right)\right]-6\left[X^2\left(\tfrac{1}{\sqrt2}\right)\left(\tfrac{1}{\sqrt2}\right)-Y^2\left(\tfrac{1}{\sqrt2}\right)\left(\tfrac{1}{\sqrt2}\right)\right]+5\left[X^2\left(\tfrac12\right)+Y^2\left(\tfrac12\right)\right]-8=0$$

$$\Rightarrow \frac52X^2+\frac52Y^2-\frac62X^2+\frac62Y^2+\frac{5X^2}{2}+\frac{5Y^2}{2}-8=0$$

Multiplying by 2, we have

$$5X^2+5Y^2-6X^2+6Y^2+5X^2+5Y^2-16=0$$

$$4X^2+16Y^2-16=0$$

$$\boxed{X^2+4Y^2-4=0} \quad \because 4\neq0$$

General Equation of Conics :- (With Rotation)

The second degree equation

$$ax^2+by^2+2hxy+2gx+2fy+c=0 \quad \text{—(1)}$$

is called General Equation of Conics.

In this Equation

h^2-ab is called discriminant of Conics Equation.

If

(i) $h^2-ab<0$ then Conics is circle or ellipse.

(ii) $h^2-ab=0$ then Conics is parabola.

(iii) $h^2-ab>0$ then Conics is hyperbola.

This equation can be transformed into rotated plane as:

$$AX^2+BY^2+2GX+2FY+C=0$$

with $2\theta = \tan^{-1}\left(\frac{2h}{a-b}\right)$

for $0^\circ<\theta\leq 90^\circ$

* If $a=b$ or $a=0=b$ then $\theta = 45^\circ$

If $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$ then Equation (1) represents a pair of Straight Lines.

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