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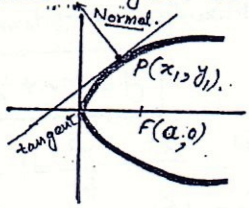
M.Sc Math
0345-6510779

Equation of tangent and Normal for:

(i) Parabola:-

Let (x_1, y_1) be any arbitrary Point lying on parabola $y^2 = 4ax$.



Diff. w.r.t. x

$$2y\frac{dy}{dx} = 4a$$

$$\frac{dy}{dx} = \frac{2a}{y}$$

$$\left.\frac{dy}{dx}\right|_{(x_1,y_1)} = \frac{2a}{y_1} = \text{Slope of tangent at } (x_1, y_1)$$

Equation of tangent through (x_1, y_1)

$$(y - y_1) = \left.\frac{dy}{dx}\right|_p (x - x_1)$$

$$(y - y_1) = \frac{2a}{y_1}(x - x_1)$$

$$yy_1 - y_1^2 = 2ax - 2ax_1$$

$$yy_1 - 2ax = y_1^2 - 2ax_1$$

$$\because y_1^2 = 4ax_1 \text{ at } (x_1, y_1)$$

$$yy_1 - 2ax = 4ax_1 - 2ax_1$$

$$yy_1 - 2ax = 2ax_1$$

$$yy_1 = 2ax + 2ax_1$$

$$\boxed{yy_1 = 2a(x + x_1)}$$

$$\text{Slope of Normal} = \frac{-1}{\left.\frac{dy}{dx}\right|_p} = \frac{-y_1}{2a}$$

Equation of Normal:

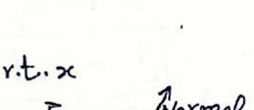
$$\boxed{y - y_1 = \frac{-y_1}{2a}(x - x_1)}$$

(ii) Ellipse:-

Let (x_1, y_1) be a point on ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Differentiating w.r.t. x



$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\frac{y}{b^2}\frac{dy}{dx} = -\frac{x}{a^2}$$

$$\left.\frac{dy}{dx}\right]_p = -\frac{b^2 x_1}{a^2 y_1}$$

Equation of tangent through (x_1, y_1)

$$(y - y_1) = \left.\frac{dy}{dx}\right|_p (x - x_1)$$

$$y - y_1 = \frac{-b^2 x_1}{a^2 y_1}(x - x_1)$$

$$\frac{yy_1 - y_1^2}{b^2} = \frac{-xx_1 + x_1^2}{a^2}$$

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$$

$$\because \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1 \quad \text{at } (x_1, y_1)$$

$$\Rightarrow \boxed{\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1}$$

$$\text{Slope of normal} = \frac{-1}{\left.\frac{dy}{dx}\right]_p} = \frac{a^2 y_1}{b^2 x_1}$$

Equation of normal is

$$y - y_1 = \frac{a^2 y_1}{b^2 x_1}(x - x_1)$$

$$b^2 x_1 y - b^2 x_1 y_1 = a^2 x y_1 - a^2 x_1 y_1$$

$$a^2 x_1 y_1 - b^2 x_1 y_1 = a^2 x y_1 - b^2 x_1 y$$

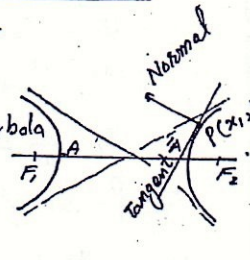
$$(a^2 - b^2) x_1 y_1 = a^2 x y_1 - b^2 x_1 y$$

$$\boxed{a^2 - b^2 = \frac{a^2 x}{x_1} - \frac{b^2 y}{y_1}}$$

(iii) Hyperbola:-

Let $P(x_1, y_1)$ be any point on hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$



Diff. w.r.t. "x"

$$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 1 \Rightarrow \frac{y}{b^2}\frac{dy}{dx} = \frac{x}{a^2}$$

$$\left.\frac{dy}{dx}\right]_p = \frac{b^2 x_1}{a^2 y_1}$$

Equation of tangent through (x_1, y_1)

$$y - y_1 = \frac{b^2 x_1}{a^2 y_1}(x - x_1)$$

$$a^2 yy_1 - a^2 y_1^2 = b^2 xx_1 - b^2 x_1^2$$

$$b^2 x_1^2 - a^2 y_1^2 = b^2 xx_1 - a^2 yy_1$$

Dividing by $a^2 b^2$

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$$\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}=\frac{xx_1}{a^2}-\frac{yy_1}{b^2}$$

$\because \frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}=1$ at (x_1, y_1)

So $\boxed{\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1}$

Slope of Normal $= \frac{-a^2y_1}{b^2x_1}$

Equation of normal line is

$$y-y_1=\frac{-a^2y_1}{b^2x_1}(x-x_1)$$

$$b^2x_1y-b^2x_1y_1=-a^2xy_1+a^2x_1y_1$$

$$a^2xy_1+b^2x_1y=a^2x_1y_1+b^2x_1y_1$$

$$a^2xy_1+b^2x_1y=(a^2+b^2)x_1y_1$$

$$\boxed{\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2}$$

Theorem:- Find the condition that $y=mx+c$ is tangent to

(i) $y^2=4ax$

(ii) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

(iii) $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

Proof:- (i) $y^2=4ax$ —— ①

$y=mx+c$ —— ②

① and ②

$\Rightarrow (mx+c)^2=4ax$

$$m^2x^2+c^2+2mxc-4ax=0$$

$$m^2x^2+(2mc-4a)x+c^2=0$$

$y=mx+c$ will be tangent to ① if Disc $=0$

So $(2mc-4a)^2-4(m^2)(c^2)=0$

$$\cancel{4m^2c^2}+16a^2-16amc-\cancel{4m^2c^2}=0$$

$$16a[a-mc]=0$$

$\because 16a\neq 0 \Rightarrow a-mc=0$

$c=\frac{a}{m}$ which is required

Now Equation of tangent is

$$\boxed{y=mx+\frac{a}{m}}$$

(ii) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ —— ③

① and ③

$$\Rightarrow \frac{x^2}{a^2}+\frac{(mx+c)^2}{b^2}=1$$

$$\Rightarrow b^2x^2+(m^2x^2+c^2+2mxc)a^2=a^2b^2$$

$$\Rightarrow (a^2m^2+b^2)x^2+(2mca^2)x+(a^2c^2-a^2b^2)=0$$

For $y=mx+c$ being tangent to ③

Disc $=0$

$$\Rightarrow (2mca^2)^2-4(a^2m^2+b^2)(a^2c^2-a^2b^2)=0$$

$$4m^2c^2a^4-4[a^4m^2c^2-a^4m^2b^2+b^2a^2c^2-a^2b^4]=0$$

$$\cancel{4m^2c^2a^4}-\cancel{4a^4m^2c^2}+4a^4m^2b^2-4a^2b^2c^2+4a^2b^4=0$$

$$4a^2b^2c^2=4a^4m^2b^2+4a^2b^4$$

$$4a^2b^2c^2=4a^2b^2[m^2a^2+b^2]$$

$$c^2=m^2a^2+b^2$$

$$c=\pm\sqrt{m^2a^2+b^2}$$

is required Condition.

So Equation of tangent

$$\boxed{y=mx\pm\sqrt{m^2a^2+b^2}}$$

(iii) Now replacing "b" by "$-b$"

$$c=\pm\sqrt{m^2a^2-b^2}$$

is required Condition.

So equation of tangent becomes:

$$\boxed{y=mx\pm\sqrt{m^2a^2-b^2}}$$

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Theorem:- Show that product of the distances from the foci to any tangent to hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is Constant.

Proof:- The line $y=mx\pm\sqrt{a^2m^2-b^2}$ is the tangent to hyperbola.

$F_1(c,0)$ and $F_2(-c,0)$ are foci of hyperbola.

Let d_1 is the distant of $F_1(c,0)$

From $y = mx \pm \sqrt{a^2m^2-b^2}$

then

$$d_1 = \frac{|mc \pm \sqrt{a^2m^2-b^2}|}{\sqrt{1+m^2}} \quad \text{---}(1)$$

Let d_2 is the distance of $F_2(-c,0)$

from $y = mx \pm \sqrt{a^2m^2-b^2}$

then

$$d_2 = \frac{|-mc \pm \sqrt{a^2m^2-b^2}|}{\sqrt{1+m^2}} \quad \text{---}(2)$$

Now

$$d_1 d_2 = \frac{|mc \pm \sqrt{a^2m^2-b^2}|}{\sqrt{1+m^2}} \times \frac{|-mc \pm \sqrt{a^2m^2-b^2}|}{\sqrt{1+m^2}}$$

$$d_1 d_2 = \frac{|(\pm\sqrt{a^2m^2-b^2})^2 - (mc)^2|}{1+m^2}$$

$$d_1 d_2 = \frac{|a^2m^2-b^2-m^2c^2|}{1+m^2}$$

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$\because c^2 = a^2+b^2 \Rightarrow b^2 = c^2-a^2$

$$d_1 d_2 = \frac{|a^2m^2-c^2+a^2-m^2c^2|}{1+m^2}$$

$$d_1 d_2 = \frac{|a^2(1+m^2)-c^2(1+m^2)|}{(1+m^2)}$$

$$d_1 d_2 = \frac{|a^2-c^2|\cancel{|1+m^2|}}{\cancel{(1+m^2)}}$$

$$d_1 d_2 = |a^2-c^2| = |-b^2|$$

$d_1 d_2 = b^2 =$ Constant (Proved)

Exercise: 6.7

Q.1 Find the equation of tangent and normal at given point:

Sol:- (i) $y^2 = 4ax$ at $(at^2, 2at)$

Diff. w.r.t. x

$$2y\frac{dy}{dx} = 4a \Rightarrow \frac{dy}{dx} = \frac{2a}{y}$$

$$\left.\frac{dy}{dx}\right]_{(at^2,2at)} = \frac{2a}{2at} = \frac{1}{t}$$

$$\left.\frac{dy}{dx}\right]_p = \frac{1}{t}$$

Equation of tangent is

$$y-y_1 = \left.\frac{dy}{dx}\right]_p (x-x_1)$$

$$y-2at = \frac{1}{t}(x-at^2)$$

$$yt-2at^2 = x-at^2$$

$$x-at^2+2at^2-yt=0$$

$\boxed{x-yt+at^2=0}$ is equation of tangent.

Slope of normal $= \dfrac{-1}{\left.\frac{dy}{dx}\right]_p} = \dfrac{-1}{(1/t)} = -t$

Equation of normal:

$$y-y_1 = \frac{-1}{\left.\frac{dy}{dx}\right]_p}(x-x_1)$$

$$(y-2at) = -t(x-at^2)$$

$$y-2at = -xt+at^3$$

$\boxed{xt+y-2at-at^3=0}$

is Equation of normal.

(ii) $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ at $(a\cos\theta, b\sin\theta)$

$$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$$

$$\frac{dy}{dx} = \frac{-b^2x}{a^2y} \Rightarrow \left.\frac{dy}{dx}\right]_p = \frac{-b^2 a\cos\theta}{a^2 b\sin\theta}$$

$$\left.\frac{dy}{dx}\right]_p = -\frac{b\cos\theta}{a\sin\theta}$$

Equation of tangent

$$y-y_1 = \left.\frac{dy}{dx}\right]_p(x-x_1)$$

$$y-b\sin\theta = \frac{-b\cos\theta}{a\sin\theta}(x-a\cos\theta)$$

$$ay\sin\theta - ab\sin^2\theta = -bx\cos\theta + ab\cos^2\theta$$

$$ay\sin\theta + bx\cos\theta = ab\cos^2\theta + ab\sin^2\theta$$

$$ay\sin\theta + bx\cos\theta = ab(\cos^2\theta+\sin^2\theta)$$

$\because \sin^2\theta+\cos^2\theta=1$

$\Rightarrow \boxed{\dfrac{y}{b}\sin\theta + \dfrac{x}{a}\cos\theta = 1}$ Eq. of tangent.

Slope of normal $= \dfrac{-1}{\left.\frac{dy}{dx}\right]_p} = \dfrac{a^2 b\sin\theta}{b^2 a\cos\theta}$

$$= \frac{a\sin\theta}{b\cos\theta}$$

Eq of normal

$$y-b\sin\theta = \frac{a\sin\theta}{b\cos\theta}(x-a\cos\theta)$$

$$yb\cos\theta - b^2\sin\theta\cos\theta = ax\sin\theta - a^2\sin\theta\cos\theta$$

$$(a^2-b^2)\sin\theta\cos\theta = ax\sin\theta - by\cos\theta$$

$$\Rightarrow \frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2-b^2$$

$\Rightarrow \boxed{ax\sec\theta - by\csc\theta = a^2-b^2}$ Normal Equation.

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(iii) $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at $P(a\sec\theta, b\tan\theta)$

Diff. w.r.t. x

$$\frac{2x}{a^2}-\frac{2y}{b^2}\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=\frac{b^2x}{a^2y}$$

$$\left.\frac{dy}{dx}\right]_P=\frac{b^2a\sec\theta}{a^2b\tan\theta}=\frac{b\sec\theta}{a\tan\theta}$$

Equation of tangent is:

$$y-b\tan\theta=\frac{b\sec\theta}{a\tan\theta}(x-a\sec\theta)$$

$$ay\tan\theta-ab\tan^2\theta=xb\sec\theta-ab\sec^2\theta$$

$$ab\sec^2\theta-ab\tan^2\theta=xb\sec\theta-ay\tan\theta$$

$$ab[\sec^2\theta-\tan^2\theta]=xb\sec\theta-ay\tan\theta.$$

$$\because \sec^2\theta-\tan^2\theta=1$$

$$ab=xb\sec\theta-ay\tan\theta$$

$$\boxed{1=\frac{x}{a}\sec\theta-\frac{y}{b}\tan\theta}$$

Slope of normal $=\frac{-1}{\left.\frac{dy}{dx}\right]_P}=\frac{-a\tan\theta}{b\sec\theta}$

Eq. of normal:

$$y-b\tan\theta=\frac{-a\tan\theta}{b\sec\theta}(x-a\sec\theta)$$

$$yb\sec\theta-b^2\sec\theta\tan\theta=-ax\tan\theta+a^2\sec\theta\tan\theta$$

$$ax\tan\theta+yb\sec\theta=(a^2+b^2)\tan\theta\sec\theta$$

$$\frac{ax\tan\theta}{\tan\theta\sec\theta}+\frac{yb\sec\theta}{\tan\theta\sec\theta}=a^2+b^2$$

$$\frac{ax}{\sec\theta}+\frac{yb}{\tan\theta}=a^2+b^2$$

$$\boxed{ax\cos\theta+yb\cot\theta=a^2+b^2}$$

Q.2 Write equation of tangent:

(i) $3x^2=-16y$ at $y=3$

$$\Rightarrow 3x^2=-16(-3)=48$$

$$x^2=16 \Rightarrow x=\pm4$$

So points are $(4,-3),(-4,-3)$

Now by diff. w.r.t. x

$$6x=-16\frac{dy}{dx} \Rightarrow \frac{dy}{dx}=\frac{-3x}{8}$$

$$\left.\frac{dy}{dx}\right]_{(4,-3)}=\frac{-3(4)}{8}=\frac{-3}{2}$$

Eq of tangent at $(4,-3)$ is

$$y+3=\frac{-3}{2}(x-4)$$

$$2y+6=-3x+12$$

$$\boxed{3x+2y-6=0}$$

$$\left.\frac{dy}{dx}\right]_{(-4,-3)}=\frac{-3(-4)}{8}=\frac{3}{2}$$

Eq of tangent at $(-4,-3)$

$$y+3=\frac{3}{2}(x+4)$$

$$2y+6=3x+12$$

$$\boxed{3x-2y+6=0}$$

(ii) $3x^2-7y^2=20$ at $y=-1$

$$\Rightarrow 3x^2-7(-1)^2=20 \Rightarrow 3x^2-7=20$$

$$\Rightarrow 3x^2=27 \Rightarrow x^2=9 \Rightarrow x=\pm3$$

Points are $(3,-1),(-3,-1)$

Now by diff. w.r.t. x

$$6x-14y\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=\frac{3}{7}\frac{x}{y}$$

$$\left.\frac{dy}{dx}\right]_{(3,-1)}=\frac{3(3)}{7(-1)}=\frac{-9}{7}$$

Eq of tangent at $(3,-1)$ is:

$$y+1=\frac{-9}{7}(x-3)$$

$$7y+7=-9x+27$$

$$\boxed{9x+7y-20=0}$$

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$$\left.\frac{dy}{dx}\right]_{(-3,-1)}=\frac{3(-3)}{7(-1)}=\frac{9}{7}$$

Eq of tangent at $(-3,-1)$ is:

$$y+1=\frac{9}{7}(x+3)$$

$$7y+7=9x+27$$

$$\boxed{9x-7y+20=0}$$

(iii) $3x^2-7y^2+2x-y-48=0$ at $x=4$

$$\Rightarrow 3(4)^2-7y^2+2(4)-y-48=0$$

$$\Rightarrow 48-7y^2+8-y-48=0 \Rightarrow 7y^2+y-8=0$$

$$\Rightarrow 7y^2+8y-7y-8=0 \Rightarrow (y-1)(7y+8)=0$$

$$y=1 \text{ and } y=-8/7$$

Points are $(4,1)$ and $(4,-8/7)$

Diff. w.r.t. "x"

$$6x-14y\frac{dy}{dx}+2-\frac{dy}{dx}=0$$

$$(6x+2)=(14y+1)\frac{dy}{dx} \Rightarrow \frac{dy}{dx}=\frac{6x+2}{14y+1}$$

$$\left.\frac{dy}{dx}\right]_{(4,1)}=\frac{6(4)+2}{14(1)+1}=\frac{26}{15}$$

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Eq of tangent through (4,1)

$y-1 = \frac{26}{15}(x-4)$

$15y-15 = 26x-104$

$26x-15y-104+15=0$

$\boxed{26x-15y-89=0}$

$\left.\frac{dy}{dx}\right]_{(4,-8/7)} = \frac{6(4)+2}{14(-8/7)+1} = \frac{26}{(-105/7)}$

$= \frac{-26\times 7}{105} = \frac{-182}{105} = \frac{-26}{15}$

Eq of tangent through $(4,-8/7)$

$y+\frac{8}{7} = \frac{-26}{15}(x-4)$

$15y+\frac{120}{7} = -26x+104$

$26x+15y+\frac{120}{7}-104=0$

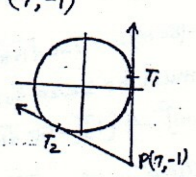
$26x+15y-\frac{608}{7}=0$

$\boxed{13x+\frac{15}{2}y-\frac{304}{7}=0}$

Q3 Find Equations of tangent:

(i) $x^2+y^2=25$ through $(7,-1)$

$\because$ Equation of tangent to $x^2+y^2=a^2$ is



$y = mx + a\sqrt{1+m^2}$

$\because a^2=25 \Rightarrow a=5$

$y = mx+5\sqrt{1+m^2}$

$\because (7,-1)$ lies on it so

$-1 = 7m+5\sqrt{1+m^2}$

$\Rightarrow 5\sqrt{1+m^2} = -1-7m$

Squaring both sides

$\Rightarrow 25+25m^2 = 1+49m^2+14m$

$\Rightarrow 24m^2+14m-24=0$

$\Rightarrow 12m^2+7m-12=0 \quad \because 2\neq 0$

$\Rightarrow 12m^2+16m-9m-12=0$

$\Rightarrow 4m(3m+4)-3(3m+4)=0$

$\Rightarrow (4m-3)(3m+4)=0$

$\Rightarrow m=3/4$ and $m=-4/3$

So Equations are:

$y = \frac{3}{4}x+5\sqrt{1+\frac{9}{16}}$ and $y=\frac{-4}{3}x+5\sqrt{1+\frac{16}{9}}$

$y=\frac{3}{4}x+5\sqrt{\frac{25}{16}}$ and $y=\frac{-4}{3}x+5\sqrt{\frac{25}{9}}$

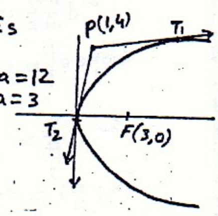
$y=\frac{3}{4}x+\frac{25}{4}$ and $y=-\frac{4x}{3}+\frac{25}{3}$

$4y=3x+25$ and $3y=-4x+25$

$\boxed{3x-4y+25=0}$ and $\boxed{4x+3y-25=0}$

(ii) $y^2=12x$ through (1,4)

$\because$ Eq of tangent to $y^2=4ax$ is



$y = mx+\frac{a}{m}$ where $4a=12$, $a=3$

$y = mx+\frac{3}{m}$

$\because$ (1,4) lies on tangent so

$4 = m+\frac{3}{m} \Rightarrow 4m = m^2+3$

$\Rightarrow m^2-4m+3=0$

$(m-3)(m-1)=0$

$m-3=0$ and $m-1=0$

$m=3$ and $m=1$

So equations are:

$y=3x+\frac{3}{3}$ and $y=x+3$

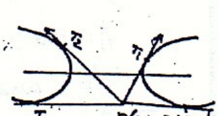
$y=3x+1$ and $y=x+3$

$\boxed{3x-y+1=0}$ and $\boxed{x-y+3=0}$

(iii) $x^2-2y^2=2$ through $(1,-2)$

or $\frac{x^2}{(\sqrt{2})^2}-\frac{y^2}{1^2}=1$ through $(1,-2)$

$\because$ Eq of tangent to $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$



is $y = mx\pm\sqrt{m^2a^2-b^2}$

where $a^2=2$ and $b^2=1$

$y = mx\pm\sqrt{2m^2-1}$

$y-mx = \pm\sqrt{2m^2-1}$

$\because (1,-2)$ lies on tangent so $-2-m = \pm\sqrt{2m^2-1}$

Squaring both Sides

$4+m^2+4m = 2m^2-1 \Rightarrow m^2-4m-5=0$

$(m-5)(m+1)=0 \Rightarrow m=5,-1$

So Equations are:

$y=5x\pm\sqrt{2(25)-1}$ and $y=-x\pm\sqrt{2-1}$

$y=5x\pm 7$ and $y=-x\pm 1$

$\boxed{5x-y+7=0}$, $\boxed{5x-y-7=0}$, $\boxed{x+y+1=0}$, $\boxed{x+y-1=0}$

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Q.4 $y^2 = 8x$ and $2x+3y=10$

$3y = -2x+10$

$y = \frac{-2}{3}x + \frac{10}{3}$

Slope of line $= -2/3$

$\because$ Normal is parallel to line so

Slope of normal $= -2/3$

Now $y^2 = 8x$

$2y\frac{dy}{dx} = 8 \Rightarrow \frac{dy}{dx} = \frac{4}{y}$

Slope of normal $= \frac{-1}{\frac{dy}{dx}} = \frac{-y}{4}$

so $-\frac{y}{4} = -\frac{2}{3} \Rightarrow y = 8/3$

So $\left(\frac{8}{3}\right)^2 = 8x \Rightarrow x = 8/9$

So point is $\left(\frac{8}{9}, \frac{8}{3}\right)$

Equation of normal is

$y - 8/3 = -\frac{2}{3}\left(x - \frac{8}{9}\right)$

$y - \frac{8}{3} = -\frac{2x}{3} + \frac{16}{27}$

$27y - 72 = -18x + 16$

$\boxed{18x + 27y - 88 = 0}$

Q.5 $\frac{x^2}{4} + y^2 = 1$ and $2x - 4y + 5 = 0$

$\Rightarrow 4y = 2x+5 \Rightarrow y = \frac{1}{2}x + 5/4$

Slope of line = Slope of tangent $= 1/2$

Equations of tangent to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$y = mx \pm \sqrt{a^2m^2 + b^2}$

$\because a^2 = 4$ and $b^2 = 1$

$\Rightarrow y = \frac{1}{2}x \pm \sqrt{4\left(\frac{1}{4}\right)+1}$

$y = \frac{1}{2}x \pm \sqrt{2} \Rightarrow 2y = x \pm 2\sqrt{2}$

or $2y = x + 2\sqrt{2}$ and $2y = x - 2\sqrt{2}$

$\boxed{x - 2y + 2\sqrt{2} = 0}$ and $\boxed{x - 2y - 2\sqrt{2} = 0}$

Q.6 $9x^2 - 4y^2 = 36$ and $5x - 2y + 7 = 0$

$\frac{x^2}{4} - \frac{y^2}{9} = 1$ and $2y = 5x+7$

$\Rightarrow y = 5/2\, x + 7/2$ so Slope of line $= 5/2$

Slope of tangent $= 5/2$

$\because$ Equations of tangent to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

are: $y = mx \pm \sqrt{a^2m^2 - b^2}$

$\Rightarrow y = \frac{5x}{2} \pm \sqrt{4\left(\frac{25}{4}\right) - 9}$ $\quad \because a^2 = 4$, $b^2 = 9$

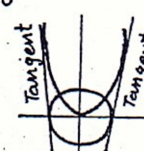
$y = \frac{5x}{2} \pm \sqrt{16} \Rightarrow y = \frac{5x}{2} \pm 4$

$2y = 5x \pm 8 \Rightarrow 2y = 5x+8$ and $2y = 5x - 8$

or $\boxed{5x - 2y + 8 = 0}$ and $\boxed{5x - 2y - 8 = 0}$

Q.7 *Find Equations of Common tangents to Conics:*

(i) $x^2 = 80y$ and $x^2 + y^2 = 81$



Tangents to both conics which are common to be determine.

Equation of tangent to $x^2 + y^2 = a^2$ is

$y = mx + a\sqrt{1+m^2}$ $\quad$ where $a^2 = 81 \Rightarrow a = 9$

$y = mx + 9\sqrt{1+m^2}$ —— (1)

Now $x^2 = 80y \Rightarrow x^2 = 80[mx + 9\sqrt{1+m^2}]$

$x^2 - (80m)x - 720\sqrt{1+m^2} = 0$

Disc. must be zero for $y = mx + c$ being tangent so

$[-80m]^2 - 4[1][-720\sqrt{1+m^2}] = 0$

$640m^2 + 2880\sqrt{1+m^2} = 0$

$2m^2 + 9\sqrt{1+m^2} = 0 \Rightarrow 2m^2 = -9\sqrt{1+m^2}$

Squaring $4m^4 = 81 + 81m^2$

$\Rightarrow 4m^4 - 81m^2 - 81 = 0$

Quadratic in m^2 so $m^2 = \frac{81 \pm \sqrt{6561 + 1296}}{8}$

$m^2 = \frac{81 \pm \sqrt{7857}}{8}$

$m^2 = \frac{81 + \sqrt{7857}}{8}$ $\qquad m^2 = \frac{81 - \sqrt{7857}}{8}$ Not Possible being $-ve$

$m = \pm\sqrt{\frac{81 + \sqrt{7857}}{8}}$ $\quad$ (As square is always $+ve$)

So Equations of tangents are:

$y = \pm\frac{\sqrt{81+\sqrt{7857}}}{2\sqrt{2}} + 9\sqrt{1 + \frac{81+\sqrt{7857}}{8}}$

$\boxed{y = \pm\frac{\sqrt{81+\sqrt{7857}}}{2\sqrt{2}} + \frac{9}{2\sqrt{2}}\sqrt{89+\sqrt{7857}}}$

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(ii) $y^2=16x$ and $x^2=2y$

$\because$ Equation of tangent to $y^2=4ax$ is

$y = mx + \frac{a}{m}$ where $a=4$

$y = mx + \frac{4}{m}$

$\Rightarrow x^2 = 2\left[mx + 4/m\right]$

$\Rightarrow x^2 = 2mx + 8/m \Rightarrow mx^2 = 2m^2x + 8$

$\Rightarrow mx^2 - 2m^2x - 8 = 0$

For tangent Disc. must be zero

$4m^4 + 32m \Rightarrow 4m\left[m^3+8\right] = 0$

$4m \neq 0 \Rightarrow m^3+8=0 \Rightarrow m=-2.$

So Eq of tangent is

$y = -2x + \frac{4}{-2} \Rightarrow y = -2x-2$

$$\boxed{2x+y+2=0}$$

Q.8 Find the points of intersection:

(i) $\frac{x^2}{18} + \frac{y^2}{8} = 1$ — (1) and $\frac{x^2}{3} - \frac{y^3}{3} = 1$ — (2)

(2) $\Rightarrow x^2 - y^2 = 3$ or $y^2 = x^2 - 3$

(1) $\Rightarrow \frac{x^2}{18} + \frac{x^2-3}{8} = 1 \Rightarrow \frac{x^2}{18} + \frac{x^2}{8} = 1 + \frac{3}{8}$

$x^2\left[\frac{1}{18} + \frac{1}{8}\right] = \frac{11}{8} \Rightarrow x^2\left[\frac{8+18}{18\times 8}\right] = \frac{11}{8}$

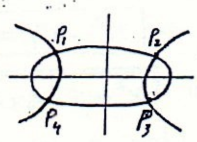
$x^2 = \frac{11\times 18}{26} \Rightarrow x^2 = \frac{99}{13}$

$x = \pm\sqrt{\frac{99}{13}}$

$y^2 = \frac{99}{13} - 3 = \frac{99-39}{13} = \frac{60}{13}$

$y = \pm\sqrt{\frac{60}{13}}$

So pts. of intersection are $\left(\pm\sqrt{\frac{99}{13}}, \pm\sqrt{\frac{60}{13}}\right)$



(ii) $x^2+y^2=8$ — (1) and $x^2-y^2=1$ — (2)

Adding (1) and (2)

$2x^2 = 9 \Rightarrow x^2 = \frac{9}{2} \Rightarrow x = \pm\frac{3}{\sqrt{2}}$

(2) $\Rightarrow y^2 = x^2 - 1 = \frac{9}{2} - 1 = \frac{7}{2} \Rightarrow y = \pm\sqrt{7/2}$

Pts are $\left(\pm\frac{3}{\sqrt{2}}, \pm\sqrt{7/2}\right)$

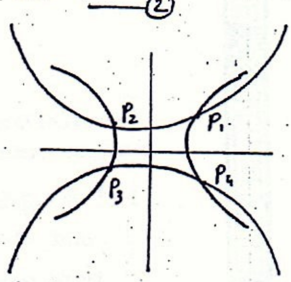
(iii) $3x^2-4y^2=12$ — (1) $3y^2-2x^2=7$ — (2)

$3\times$(2) $\Rightarrow -\cancel{6x^2}+9y^2=21$

$2\times$(1) $\Rightarrow \cancel{6x^2}-8y^2=24$

$y^2 = 45$

$y = \pm\sqrt{45}$



(2) $\Rightarrow 3(45) - 2x^2 = 7$

$\Rightarrow 135 - 7 = 2x^2$

$2x^2 = 128 \Rightarrow x^2 = 64 \Rightarrow x = \pm 8$

So pts of intersection are: $(\pm 8, \pm\sqrt{45})$

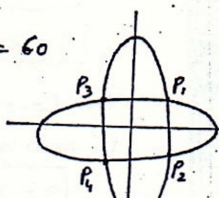
(iv) $3x^2+5y^2=60$ — (1) and $9x^2+y^2=124$ — (2)

(2) $\Rightarrow y^2 = 124 - 9x^2$

(1) $\Rightarrow 3x^2 + 5(124 - 9x^2) = 60$

$3x^2 + 620 - 45x^2 = 60$

$-42x^2 = -560$



$x^2 = \frac{-560}{-42} = \frac{40}{3}$

$x = \pm\frac{2\sqrt{10}}{\sqrt{3}} \Rightarrow y^2 = 124 - 9\left(\frac{40}{3}\right)$

$y^2 = \frac{372-360}{3} = \frac{12}{3} = 4 \Rightarrow y = \pm 2$

So pts of intersection are:

$\left(\pm 2\sqrt{10/3}, \pm 2\right)$

(v) $4x^2+y^2=16$ — (1) $x^2+y^2+y+8=0$

$\Rightarrow \frac{x^2}{4} + \frac{y^2}{16} = 1$ $x^2 + y^2 + y = -8$ — (2)

$(x-0)^2 + y^2 + y + 1/4 = -8 + 1/4$

$(x-0)^2 + (y+1/2)^2 = \frac{-31}{4}$

which is impossible because

$r^2 \neq \frac{-31}{4}$ Must be +ve

So there is no point of intersection.

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