TAHIR MEHMOOD
M.Sc Math
0345-6510779

TANGENT:-

"A tangent to any curve is a line which touches the curve externally at just one point."

NORMAL:-

"A line perpendicular to the tangent is called Normal"

* For any function $y=f(x)$ or $f(x,y)=0$, $\left.\frac{dy}{dx}\right]_p$ is called Slope of the tangent and $-1/(dy/dx)_p$ is called Slope of Normal at any point "P"

Position of a Point w.r.t. Circle:-

Let $x^2+y^2+2gx+2fy+c=0$ be a circle and $P(x_1, y_1)$ be any point then

(i) Point lies on the Circle if $x_1^2+y_1^2+2gx_1+2fy_1+c=0$

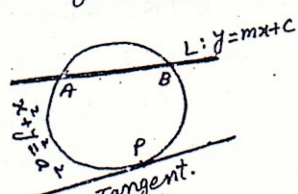
(ii) Point lies out side the circle if $x_1^2+y_1^2+2gx_1+2fy_1+c>0$

(iii) Point lies inside the circle if $x_1^2+y_1^2+2gx_1+2fy_1+c<0$

THEOREM

The Line $y=mx+c$ intersects the circle $x^2+y^2=a^2$ in at most two points.

Proof:. Given that $y=mx+c$ —① and $x^2+y^2=a^2$ —②



$\Rightarrow \quad x^2+(mx+c)^2=a^2$

$\Rightarrow \quad x^2+m^2x^2+2mxc+c^2-a^2=0$

$\Rightarrow (1+m^2)x^2+(2mc)x+(c^2-a^2)=0$ which is quadratic Equation so it has at most two roots so line intersects the Circle at most two points.

Discriminant $B^2-4AC=(2mc)^2-4(1+m^2)(c^2-a^2)$

$=\cancel{4m^2c^2}-4c^2+4a^2-\cancel{4m^2c^2}+4m^2a^2=4[a^2(1+m^2)-c^2]$

The points are

(i) Real and distinct if $a^2(1+m^2)-c^2>0$

(ii) Real and Coincident if $a^2(1+m^2)-c^2=0$

(iii) Imaginary if $a^2(1+m^2)-c^2<0$

$y=mx+c$ will be tangent to $x^2+y^2=a^2$ if $a^2(1+m^2)-c^2=0$

$\Rightarrow c^2=a^2(1+m^2)$ or $c=\pm a\sqrt{1+m^2}$

So Eq. of tangent is $y=mx\pm a\sqrt{1+m^2}$.

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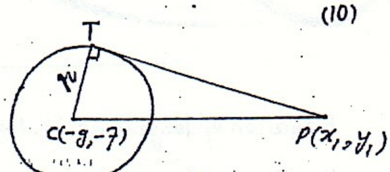
TANGENTIAL DISTANCE :-



Let $P(x_1, y_1)$ be any point outside the circle

$$x^2+y^2+2gx+2fy+c=0$$ having centre $C(-g,-f)$ and radius $|CT| = \sqrt{g^2+f^2-c}$.

From Right triangle PCT $\quad |PC|^2 = |CT|^2 + |PT|^2$

Length of tangent $|PT| = \sqrt{|CP|^2 - |CT|^2}$

$$\text{Tengential length} = \sqrt{(x_1+g)^2+(y_1+f)^2-(g^2+f^2-c)}$$

$$\text{Tengential length} = \sqrt{x_1^2+g^2+2gx_1+y_1^2+f^2+2y_1f-g^2-f^2+c}$$

$$\text{Tengential Length} = \sqrt{x_1^2+y_1^2+2gx_1+2fy_1+c}$$

EXERCISE: 6·2

Q.1 Find Eq of tangent and normal:

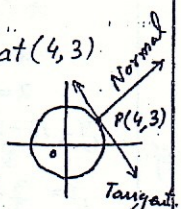
(i) $x^2+y^2=25$ at $(4,3)$

Diff. w.r.t. x

$$2x+2y\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx} = \frac{-x}{y}$$

$$\left.\frac{dy}{dx}\right|_{(4,3)} = -\frac{4}{3}$$

Equation of tangent at $(4,3)$



$$y-3 = -\frac{4}{3}(x-4)$$

$$3y-9 = -4x+16$$

$$\boxed{4x+3y-25=0}$$

Equation of Normal at $(4,3)$

$$y-3 = \frac{-1}{(-4/3)}[x-4]$$

$$y-3 = \frac{3}{4}(x-4)$$

$$4y-12 = 3x-12$$

$$\boxed{3x-4y=0}$$

$x^2+y^2=25$ at $(5\cos\theta, 5\sin\theta)$

$$\left.\frac{dy}{dx}\right|_{(5\cos\theta,5\sin\theta)} = -\frac{5\cos\theta}{5\sin\theta} = \frac{-\cos\theta}{\sin\theta}$$

Eq of tangent at $(5\cos\theta, 5\sin\theta)$

$$y-5\sin\theta = \frac{-\cos\theta}{\sin\theta}(x-5\cos\theta)$$

$$y\sin\theta - 5\sin^2\theta = -x\cos\theta + 5\cos^2\theta$$

$$x\cos\theta + y\sin\theta = 5(\cos^2\theta+\sin^2\theta)$$

$$\boxed{x\cos\theta + y\sin\theta = 5}$$

Equation of Normal at $(5\cos\theta, 5\sin\theta)$

$$y-5\sin\theta = \frac{-1}{\left(\frac{-\cos\theta}{\sin\theta}\right)}(x-5\cos\theta)$$

$$y-5\sin\theta = \frac{\sin\theta}{\cos\theta}(x-5\cos\theta)$$

$$y\cos\theta - 5\sin\theta\cos\theta = x\sin\theta - 5\sin\theta\cos\theta$$

$$\boxed{x\sin\theta - y\cos\theta = 0}$$

(ii) $3x^2+3y^2+5x-13y+2=0$ at $p\left(1, \frac{10}{3}\right)$

Diff. w.r.t. x

$$6x+6y\frac{dy}{dx}+5-13\frac{dy}{dx}=0$$

$$(6y-13)\frac{dy}{dx} = -(6x+5)$$

$$\frac{dy}{dx} = -\frac{(6x+5)}{(6y-13)} \Rightarrow \left.\frac{dy}{dx}\right|_p = \frac{-(6+5)}{\left[6\left(\frac{10}{3}\right)-13\right]}$$

$$\left.\frac{dy}{dx}\right|_p = \frac{-11}{7}$$

Equation of tangent at $(1, 10/3)$

$y-\frac{10}{3}=\frac{-11}{7}(x-1)$

$7y-\frac{70}{3}=-11x+11$

$21y-70=-33x+33$

$\boxed{33x+21y-103=0}$

Equation of normal at $(1, 10/3)$

$y-\frac{10}{3}=\frac{-1}{\left(\frac{-11}{7}\right)}(x-1)$

$\frac{3y-10}{3}=\frac{7}{11}(x-1)$

$33y-110=21x-21$

$\boxed{21x-33y+89=0}$

Q.2 $4x^2+4y^2-16x+24y-117=0$ ——(1) $\quad x=-4$

$4(-4)^2+4y^2-16(-4)+24y-117=0$

$4y^2+24y-117+64+64=0$

$4y^2+24y+11=0$

$\Rightarrow (2y+11)(2y+1)=0$

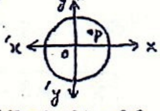
$y=\frac{-11}{2}$ and $y=-\frac{1}{2}$

Points are $(-4,-\frac{1}{2})$ and $(-4,\frac{-11}{2})$

Find Eq of tangent and normal yourself.

Q.3 Check position of point $(5,6)$

(i) $x^2+y^2=81$



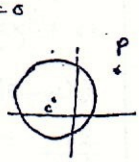
$x^2+y^2-(81)=0$

Now $(5)^2+(6)^2-81=25+36-81<0$

Point $(5,6)$ lies inside $x^2+y^2=81$

(ii) $2x^2+2y^2+12x-8y+1=0$

$x^2+y^2+6x-4y+\frac{1}{2}=0$



Now $(5)^2+(6)^2+6(5)-4(6)+\frac{1}{2}$

$=25+36+30-24+\frac{1}{2}>0$

Thus point $(5,6)$ lies outside

$2x^2+2y^2+12x-8y+11=0$

Q.4 Length of tangent from $(-5,4)$

$=\sqrt{x^2+y^2-2x+3y-\frac{131}{5}}$

$=\sqrt{(-5)^2+(4)^2-2(-5)+3(4)-\frac{131}{5}}$

$=\sqrt{25+16+10+12-\frac{131}{5}}$

$=\sqrt{63-\frac{131}{5}}=\sqrt{\frac{315-131}{5}}$

Length of tangent $=\sqrt{\frac{184}{5}}$ Ans.

Q.5 Length of Chord = ?

$2x+3y=13$ ——(1) $\qquad x^2+y^2=26$ ——(2)

$\Rightarrow 3y=13-2x$ $\qquad$ Putting in (2)

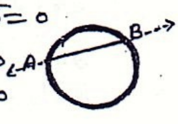
$y=\frac{13-2x}{3}$ $\qquad x^2+\left(\frac{13-2x}{3}\right)^2=26$

$x^2+\frac{169+4x^2-52x}{9}=26$

$\Rightarrow 9x^2+4x^2+169-52x=234$

$\Rightarrow 13x^2-52x-65=0$

$\Rightarrow x^2-4x-5=0$



$\Rightarrow (x-5)(x+1)=0$

$x-5=0$ and $x+1=0$

$x=5$	$x=-1$
$y=\frac{13-2(5)}{3}$	$y=\frac{13-2(-1)}{3}$
$y=\frac{3}{3}=1$	$y=\frac{15}{3}=5$

Points are $A(5,1)$ and $B(-1,5)$

Length of Chord $|AB|=\sqrt{(5+1)^2+(1-5)^2}$

$=\sqrt{(6)^2+(-4)^2}$

$=\sqrt{36+16}=\sqrt{52}$

$=2\sqrt{13}$ Ans.

Q.6 Points of intersection = ?

$x+2y=6$ ——(1) $\qquad x^2+y^2-2x-2y-39=0$ ——(2)

$\Rightarrow x=6-2y$ $\qquad$ Putting in (2), we have

$\Rightarrow (6-2y)^2+y^2-2(6-2y)-2y-39=0$

$\Rightarrow 36+4y^2-24y+y^2-12+4y-2y-39=0$

$\Rightarrow 5y^2-22y-15=0$

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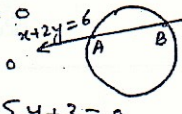
$\Rightarrow 5y^2 - 22y - 15 = 0$

$\Rightarrow (5y+3)(y-5) = 0$



$\Rightarrow y-5=0$ and $5y+3=0$

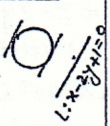
$y = 5$
$x = 6-2(5)$
$= 6-10$
$x = -4$

$y = -3/5$
$x = 6-2(-3/5)$
$x = \frac{30+6}{5} = \frac{36}{5}$

Points of intersections are

$(-4, 5), \left(\frac{36}{5}, \frac{-3}{5}\right)$

Q.7 Circle: $x^2+y^2 = 2$



Comparing with $x^2+y^2 = a^2$

(i) $a = \sqrt{2}$ and $m =$ Slope of line $x-2y+1=0$

$m = \frac{1}{2} =$ Slope of tangent

$\because c = \pm a\sqrt{1+m^2}$

$c = \pm\sqrt{2}\sqrt{1+\frac{1}{4}} = \pm\sqrt{2}\sqrt{5/4}$

$c = \pm\frac{\sqrt{10}}{2}$

Equations of tangent

$y = mx + c$

$y = \frac{1}{2}x \pm \frac{\sqrt{10}}{2} \Rightarrow 2y = x \pm \sqrt{10}$

$\boxed{x-2y+\sqrt{10}=0}$ and $\boxed{x-2y-\sqrt{10}=0}$

which are equations of tangents parallel to $x-2y+1=0$

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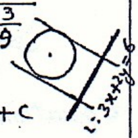
(ii) Slope of $3x+2y=6$ is $-3/2$

Slope of tangent $(m) = \frac{2}{3}$

Thus $c = \pm a\sqrt{1+m^2}$

$c = \pm\sqrt{2}\sqrt{1+\frac{4}{9}} = \pm\sqrt{2}\sqrt{\frac{13}{9}}$

$c = \pm\frac{\sqrt{26}}{3}$



Eq's of tangent $y = mx + c$

$y = \frac{2}{3}x \pm \frac{\sqrt{26}}{3} \Rightarrow 3y = 2x \pm \sqrt{26}$

Eqs of tangent perpendicular to $3x+2y=6$

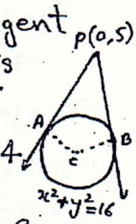
$\boxed{2x-3y+\sqrt{26}=0}$ and $\boxed{2x-3y-\sqrt{26}=0}$

Q.8 Find Eqs of tangents and points of Contact. (12)

(i) From $(0,5)$ to $x^2+y^2=16$ —①

We know that any tangent $y = mx+c$ on $x^2+y^2=a^2$ is

$y = mx + a\sqrt{1+m^2}$



From given equation $a = 4$

$\Rightarrow y - mx = 4\sqrt{1+m^2}$

Squaring both sides, we have

$y^2 - 2mxy + m^2x^2 = 16 + 16m^2$

Now, it passes through $(0,5)$ so

$y^2 = 16+16m^2 \Rightarrow 16(m^2+1) = 25$

$\Rightarrow m^2 = \frac{25}{16} - 1 \Rightarrow m^2 = 9/16$

$\Rightarrow m = \pm 3/4$

Thus Equations of tangents from $(0,5)$

$y = \pm\frac{3x}{4} + 4\sqrt{1+9/16}$

$\Rightarrow y = \pm\frac{3x}{4} + 4\cdot\frac{5}{4} = \pm\frac{3}{4}x + 5$

$\Rightarrow 4y = \pm 3x + 20$ TAHIR

$\Rightarrow 3x-4y+20=0$ and $3x+4y-20=0$

For Points of Contact

$y = \frac{3x+20}{4}$ —②

$① \Rightarrow x^2 + \left(\frac{3x+20}{4}\right)^2 = 16$

$x^2 + \frac{9x^2+400+120x}{16} = 16$

$16x^2+9x^2+400+120x = 256$

$25x^2+120x+144=0$

$\Rightarrow (5x+12)^2 = 0$

$5x+12=0$

$\Rightarrow x = -12/5$

So $y = \frac{3\left[\frac{-12}{5}\right]+20}{4}$

$y = \frac{-36+100}{20} = \frac{64}{20}$

$y = \frac{16}{5}$

$\Rightarrow \left(\frac{-12}{5}, \frac{16}{5}\right)$

$y = \frac{20-3x}{4}$ —③

$① \Rightarrow x^2 + \left(\frac{20-3x}{4}\right)^2 = 16$

$x^2 + \frac{400-120x+9x^2}{16} = 16$

$16x^2+9x^2-120x+400 = 256$

$25x^2-120x+144=0$

$\Rightarrow (5x-12)^2 = 0$

$5x-12=0$

$\Rightarrow x = 12/5$

So $y = \frac{20-3\left(\frac{12}{5}\right)}{4}$

$y = \frac{100-36}{20} = \frac{64}{20}$

$y = \frac{16}{5}$

$\Rightarrow \left(\frac{12}{5}, \frac{16}{5}\right)$

Thus points of Contact are

$\left(\frac{-12}{5}, \frac{16}{5}\right), \left(\frac{12}{5}, \frac{16}{5}\right)$

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(ii) From $(-1,2)$ to $x^2+y^2+4x+2y=0$

From Equation Centre$(-2,-1)$

$\Rightarrow g=2 \quad f=1$ and $c=0$

$r=\sqrt{g^2+f^2-c}=\sqrt{4+1-0}=\sqrt{5}$

Let 'm' be the slope of tangent drawn from $(-1,2)$ so Equation of tangent is given by

$y-y_1=m(x-x_1)$

$y-2=m(x+1)$

$\Rightarrow mx-y+m+2=0$ —— (1)

Now Eq (1) is tangent to given Circle so

distance of tangent to Centre = radius.

$$\sqrt{5}=\frac{|m(-2)-(-1)+m+2|}{\sqrt{m^2+1}} \quad \because \text{Centre}=(-2,-1)$$

$$\sqrt{5}=\frac{|3-m|}{\sqrt{m^2+1}}$$

Squaring both Sides, we have

$$5=\frac{(3-m)^2}{m^2+1} \Rightarrow 5(m^2+1)=(3-m)^2$$

$\Rightarrow 5m^2+5=9+m^2-6m$

$\Rightarrow 4m^2+6m-4=0$

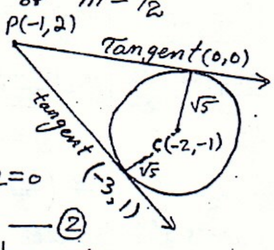
$\Rightarrow 2m^2+3m-2=0 \quad \because 2\neq 0$

$\Rightarrow 2m^2+4m-m-2=0$

$\Rightarrow (m+2)(2m-1)=0$

$\Rightarrow m+2=0$ or $2m-1=0$

$m=-2$ or $m=\frac{1}{2}$



For $m=-2$

(1) becomes

$-2x-y-2+2=0$

$2x+y=0$ —— (2)

which is tangent

To get point of Contact $y=-2x$

Putting in given Circle

$x^2+(-2x)^2+2(-2x)+4x=0$ (13)

$\Rightarrow x^2+4x^2-4x+4x=0$

$\Rightarrow 5x^2=0 \Rightarrow x=0 \Rightarrow y=2(0)=0$

Thus Contact point for $2x+y=0$ is $(0,0)$

Now consider $m=\frac{1}{2}$ in (2)

$\frac{1}{2}x-y+\frac{1}{2}+2=0$

$\Rightarrow x-2y+1+4=0 \Rightarrow x-2y+5=0$ —— (3)

which is required tangent

To get Contact point $x=2y-5$

Putting in given circle Equation.

$(2y-5)^2+y^2+4(2y-5)+2y=0$

$4y^2+25-20y+y^2+8y-20+2y=0$

$5y^2-10y+5=0 \Rightarrow y^2-2y+1=0$

$(y-1)^2=0 \Rightarrow y=1$

So $x=2-5=-3$

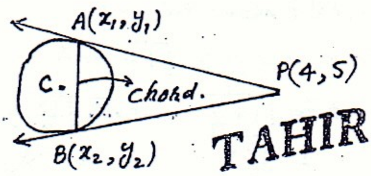
Thus Contact point is $(-3,1)$

Similarly do (iii) yourself.

Q.10 Find Equation of Chord of Contact points of tangents drawn from $(4,5)$ to the Circle

$$2x^2+2y^2-8x+12y+21=0$$

Soln:-



Equation of Circle is given by

$$2x^2+2y^2-8x+12y+21=0$$

$$\Rightarrow x^2+y^2-4x+6y+\frac{21}{2}=0$$

Comparing with general Circle Eq:

$g=-2, \ f=3, \ c=\frac{21}{2}$

Let $A(x_1,y_1)$ and $B(x_2,y_2)$ be the Contact points of tangents drawn from $P(4,5)$.

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Eq. of tangent for $A(x_1, y_1)$ is

$$xx_1 + yy_1 + g(x+x_1) + f(y+y_1) + c = 0$$

$$\Rightarrow xx_1 + yy_1 - 2(x+x_1) + 3(y+y_1) + \frac{21}{2} = 0$$

$\because$ (4,5) lies on tangent so must satisfy it.

$$\Rightarrow 4x_1 + 5y_1 - 2(4+x_1) + 3(5+y_1) + \frac{21}{2} = 0$$

$$\Rightarrow 8x_1 + 10y_1 - 4(4+x_1) + 6(5+y_1) + 21 = 0$$

$$\Rightarrow 8x_1 + 10y_1 - 16 - 4x_1 + 30 + 6y_1 + 21 = 0$$

$$\Rightarrow 4x_1 + 16y_1 + 35 = 0 \quad \text{——— (1)}$$

Now Equation of tangent for $B(x_2, y_2)$ is

$$xx_2 + yy_2 + g(x+x_2) + f(y+y_2) + c = 0$$

$$\Rightarrow xx_2 + yy_2 - 2(x+x_2) + 3(y+y_2) + \frac{21}{2} = 0$$

$\because$ (4, 5) lies on tangent so must satisfy it.

$$\Rightarrow 4x_2 + 5y_2 - 2(4+x_2) + 3(5+y_2) + \frac{21}{2} = 0$$

$$\Rightarrow 8x_2 + 10y_2 - 4(4+x_2) + 6(5+y_2) + 21 = 0$$

$$\Rightarrow 8x_2 + 10y_2 - 16 - 4x_2 + 30 + 6y_2 + 21 = 0$$

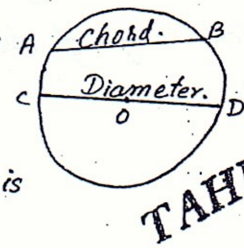
$$\Rightarrow 4x_2 + 16y_2 + 35 = 0 \quad \text{——— (2)}$$

From (1) and (2) both A and B lies on $4x + 16y + 35 = 0$
So Equation of Chord is $4x + 16y + 35 = 0$

Analytic Problems about Circle

Chord:-

A line segment that joins the two points lying on the circumference of the circle is called Chord of Circle.



Diameter:-

The greatest chord of the circle which passes through the Centre of the Circle is called diameter of the Circle.

From the Figure $\overline{AB}$ is Chord and $\overline{CD}$ is diameter.

* Length of the diameter of the Circle is twice of radius.
* Orthogonal Circles are those circles whose tangents inscribed an angle of 90°.