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Second Year

F.Sc. F.A: Part: II

Chapter: 6

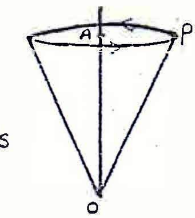
"CONIC SECTION"

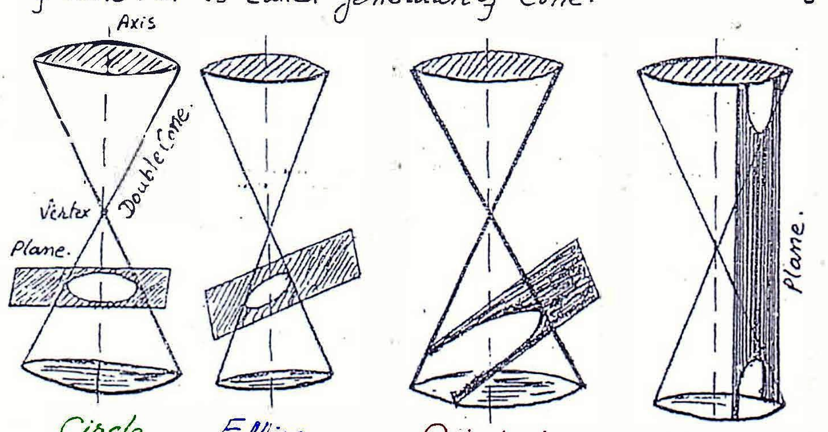
CONE :-

"The surface generated by rotating a line about a Point keeping the other end of line fixed is called Cone."

Fixed point is called vertex of cone and the line join by vertex and point about which line rotates is Called axis of Cone.

From the figure "O" is vertex and $\overline{OA}$ is axis of Cone. $\overline{OP}$ is called generator of Cone.





Circle | Ellipse | Parabola | Hyperbola

Conics are the curves obtained by cutting a double right Circular Cone by a plane. (according to above Conditions.)

These Conics are:

(i) Circle (ii) Ellipse (iii) Parabola (iv) Hyperbola

Now we discuss these Conics one by one.

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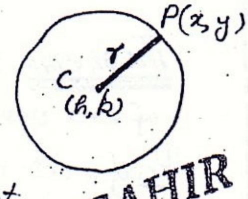
Circle:-



The set of all those Points which are equidistant From a Fixed point is called Circle.

Fixed point is called point is called Centre of circle and distance of Fixed point From any point is called radius of circle and denoted by "r".

The length of radius is called "radial distance" or "radial length" so $r = |PC|$

Equation of Circle:-

Let (h, k) be the centre and "r" be the radius of of Circle then $(x-h)^2+(y-k)^2=r^2$ is called equation of Circle (Standard Equation of Circle) Centred at (h, k) with radius "r".

Deduction:- If $C(h,k) = C(0,0)$ then $x^2+y^2=r^2$ is called Equation of Circle centred at origion.

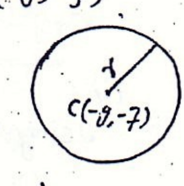
Parametric Equations of Circle $x^2+y^2=r^2$ are:

$$x = r\cos\theta \quad \wedge \quad y = r\sin\theta$$

wher $0 \le \theta \le 2\pi$

General Equation of Circle:-

The equation $x^2+y^2+2gx+2fy+c=0$ is called general Equation of Circle with centre $(-g, -f)$ and radius "$r = \sqrt{f^2+g^2-c}$"



* A Circle of radius $r<0$ is called imaginary circle.
* A circle with $r=0$ is called "Point Circle".
* A circle with $r=1$ is called "Unit Circle".
* The set of points of Circle is denoted by $S(C, r)$ so that $S(C,r) = \{ P(x,y) \mid |PC| = r \}$

Charactristics of Circle Equation:- (General)

* Circle equation is always 2nd degree equation.
* Coefficients of x^2 and y^2 are always equal in circle equation.
* Circle is independent of terms of product xy.

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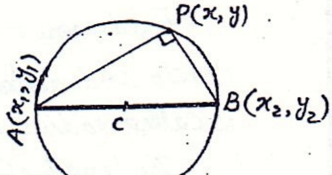
Theorem:- Equation of Circle having $A(x_1, y_1)$ and $B(x_2, y_2)$ as join of end points of diameter.



Proof:- Let $P(x, y)$ be any point on the circle having $A(x_1, y_1)$ and $B(x_2, y_2)$ as join of diameter.

(m_1) Slope of $\overline{AP} = \dfrac{y-y_1}{x-x_1}$ and (m_2) Slope of $\overline{BP} = \dfrac{y-y_2}{x-x_2}$

∵ "Angle inscribed in SemiCircle is right angle." so $m_1 \cdot m_2 = -1$

$$\Rightarrow \frac{y-y_1}{x-x_1} \times \frac{y-y_2}{x-x_2} = -1 \quad \Rightarrow (y-y_1)(y-y_2) = -(x-x_1)(x-x_2)$$

$$\Rightarrow \boxed{(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0} \quad \text{Equation of Circle}$$

EXERCISE: 6.1

Q.1 Find Equation of Circle in each case:

(i) Centre $(5, -2)$, radius $= 4$

Using Standard Equation

$(x-5)^2+(y+2)^2=(4)^2$

$x^2+25-10x+y^2+4+4y=16$

$x^2+y^2-10x+4y+25+4-16=0$

$x^2+y^2-10x+4y+13=0$

(ii) Centre $(\sqrt{2}, -3\sqrt{3})$, radius $= 2\sqrt{2}$

Using Standard Equation

$(x-\sqrt{2})^2+(y+3\sqrt{3})^2=(2\sqrt{2})^2$

$x^2+2-2\sqrt{2}x+y^2+27+6\sqrt{3}y=8$

$x^2+y^2-2\sqrt{2}x+6\sqrt{3}y+2+27-8=0$

$x^2+y^2-2\sqrt{2}x+6\sqrt{3}y+21=0$

(iii) $(-3, 2), (5, -6)$ as ends of diameter.

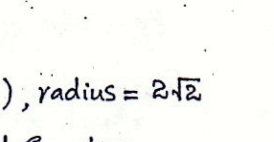
Using $(y-y_2)(y-y_1)+(x-x_1)(x-x_2)=0$

$(y-2)(y+6)+(x+3)(x-5)=0$

$y^2-2y+6y-12+x^2+3x-5x-15=0$

$x^2+y^2+3x+6y-5x-2y-27=0$

$x^2+y^2-2x+4y-27=0$ (Required)



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Q.2:- Find Centre and radius of the Circles:

(i) $x^2+y^2+12x-10y=0$

Comparing with general Eq:

$2g=12 \quad 2f=-10 \quad c=0$

$g=6 \quad f=-5 \quad c=0$

Centre $(-g,-f)=(-6,5)$

radius $(r)=\sqrt{f^2+g^2-c}$

$r=\sqrt{36+25-0}=\sqrt{61}$

Radius $=\sqrt{61}$

(iii) $x^2+y^2-6x+4y+13=0$

Comparing with general Eq:

$2g=-6 \quad 2f=4 \quad c=13$

$g=-3 \quad f=2 \quad c=13$

Centre $(-g,-f)=(3,-2)$

Radius $(r)=\sqrt{f^2+g^2-c}$

$r=\sqrt{9+4-13}=0$

Radius $=0$ (Point Circle)

(ii) $5x^2+5y^2+14x+12y-10=0$

$x^2+y^2+\frac{14}{5}x+\frac{12}{5}y-2=0$

Comparing with general Eq:

$2g=\frac{14}{5} \quad 2f=\frac{12}{5} \quad c=-2$

$g=\frac{7}{5} \quad f=\frac{6}{5} \quad c=-2$

Centre $(-g,-f)=\left(\frac{-7}{5},\frac{-6}{5}\right)$

Radius $(r)=\sqrt{f^2+g^2-c}$

$=\sqrt{\frac{36}{25}+\frac{49}{25}+2}=\sqrt{\frac{135}{25}}$

Radius $=\sqrt{\frac{27}{5}}$

(iii) $4x^2+4y^2-8x+12y-25=0$

$x^2+y^2-2x+3y-\frac{25}{4}=0$

Comparing with general Eq:

$2g=-2 \quad 2f=3 \quad c=-\frac{25}{4}$

$g=-1 \quad f=3/2 \quad c=-\frac{25}{4}$

Centre $(-g,-f)=(1,-3/2)$

Radius $(r)=\sqrt{f^2+g^2-c}$

$=\sqrt{1+\frac{9}{4}+\frac{25}{4}}=\sqrt{\frac{38}{4}}$

Radius $=\sqrt{\frac{19}{2}}$

Q.3 Find Equation of Circle through the points given below:

(i) $A(4,5)$, $B(-4,-3)$, $C(8,-3)$

Let $x^2+y^2+2gx+2fy+c=0$ be the required equation of Circle.

Now A, B, C lies on the Circle so must satisfy its equation:

$A \Rightarrow 16+25+8g+10f+c=0$; $B \Rightarrow 16+9-8g-6f+c=0$; $C \Rightarrow 64+9+16g-6f+c=0$

$8g+10f+c=-41 \quad \text{—(1)}$

$-8g-6f+c=-25 \quad \text{—(2)}$

$16g-6f+c=-73 \quad \text{—(3)}$

By (1)−(2), we have

$$\begin{aligned}8g+10f+\cancel{c}&=-41\\ \underset{+}{-}8g\ \underset{+}{-}6f\ \cancel{+c}&=\underset{+}{-}25\\ \hline 16g+16f&=-16\end{aligned}$$

$\Rightarrow g+f=-1$

$f=-1-g=1-(-2)$

$f=-1+2=1$

$\Rightarrow \boxed{f=1}$

By (2)−(3), we have

$$\begin{aligned}-8g-\cancel{6f}+\cancel{c}&=-25\\ \underset{-}{+}16g\ \cancel{\underset{+}{-}6f}\ \cancel{\underset{-}{+}c}&=\underset{+}{-}73\\ \hline -24g&=48\end{aligned}$$

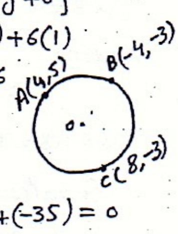
$\Rightarrow \boxed{g=-2}$

(2) $\Rightarrow c=-25+8g+6f$

$c=-25+8(-2)+6(1)$

$c=-25-16+6$

$\boxed{c=-35}$



Equation of Circle: $x^2+y^2+2(-2)x+2(1)y+(-35)=0$

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$\boxed{x^2+y^2-4x+2y-35=0}$ (Req. Equation)

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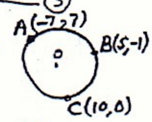
(b) $A(-7,7), B(5,-1), C(10,0)$ (5)

Let $x^2+y^2+2gx+2fy+c=0$ be the required Circle Equation.

Now A, B, C lies on the circle so must satisfy its equation:

$A \Rightarrow 49+49-14g+14f+c=0$ | $B \Rightarrow 25+1+10g-2f+c=0$ | $C \Rightarrow 100+0+20g+c=0$

$-14g+14f+c=-98$ — ① | $10g-2f+c=-26$ — ② | $c=-20g-100$ — ③



① $\Rightarrow -14g+14f-20g-100=-98$
$\Rightarrow -34g+14f=2$
$\Rightarrow -17g+7f=1$ — ④

② $\Rightarrow 10g-2f-20g-100=-26$
$\Rightarrow -10g-2f=74$
$\Rightarrow -5g-f=37 \Rightarrow f=-5g-37$ — ⑤

Putting $f=-37-5g$
$-17g+7(-37-5g)=1$
$\Rightarrow -17g-259-35g=1$
$-52g=260 \Rightarrow \boxed{g=-5}$

Now $f=-5(-5)-37$
$f=25-37=-12$
$\boxed{f=-12}$

$c=-20(-5)-100$
$c=100-100=0$
$\boxed{c=0}$

Equation of Circle: $x^2+y^2+2(-5)x+2(-12)y+(0)=0$

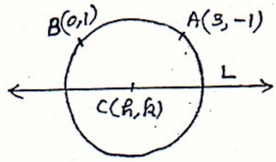
$\boxed{x^2+y^2-10x-24y=0}$ (Req. Equation)

(Similarly do remaining parts your self)

Q.4 *Find Equation of Circle in each case:*

(i) $A(3,-1), B(0,1)$ and having centre at $4x-3y-3=0$

Let $C(h,k)$ be the centre of circle:



$\Rightarrow |AC|=|BC|=$ radius

Consider $|AC|^2=|BC|^2$

$\Rightarrow (h-3)^2+(k+1)^2=(h-0)^2+(k-1)^2$

$\Rightarrow h^2+9-6h+k^2+1+2k=h^2+k^2+1-2k \Rightarrow -6h+4k=-9$ — ①

Now (h,k) lies on the line so $4h-3k=3$ — ②

①×2 $\Rightarrow -12h+8k=-18$
②×3 $\Rightarrow 12h-9k=9$
Adding $-k=-9$
$\Rightarrow \boxed{k=9}$

② $\Rightarrow 4h=3+3k=3+3(9)$
$4h=30 \Rightarrow \boxed{h=\frac{15}{2}}$

Thus Centre $C(h,k)=C(\frac{15}{2},9)$

Radius $=|BC|=\sqrt{(\frac{15}{2}-0)^2+(9-1)^2}=\sqrt{\frac{225}{4}+64}=\sqrt{\frac{481}{4}}$

Equation of Circle: $(x-15/2)^2+(y-9)^2=(\sqrt{481/4})^2$

$\Rightarrow x^2+\frac{225}{4}-15x+y^2+81-18y=\frac{481}{4} \Rightarrow x^2+y^2-15x-18y+\frac{225-481+324}{4}=0$

$\Rightarrow \boxed{x^2+y^2-15x-18y+17=0}$ (Required Equation.)

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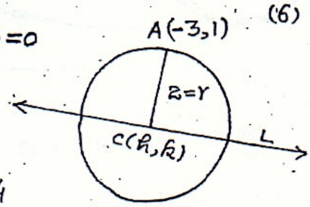
(b) $A(-3,1)$, radius $= 2$ centred at $2x-3y+3=0$



Let $C(h,k)$ be the centre of Circle:

$\Rightarrow |AC| = 2 \Rightarrow |AC|^2 = 4$

$\Rightarrow (h+3)^2+(k-1)^2=4 \Rightarrow h^2+k^2+6h-2k+10=4$

$\Rightarrow h^2+k^2+6h-2k=-6$ —— ①

Now $C(h,k)$ lies on the Line so $2h-3k=-3 \Rightarrow h = \dfrac{3k-3}{2}$ — ②

Now ① $\Rightarrow \left(\dfrac{3k-3}{2}\right)^2+(k^2)+6\left(\dfrac{3k-3}{2}\right)-2k+6=0$

$\Rightarrow \dfrac{9k^2+9-18k}{4}+k^2+\dfrac{18k-18}{2}-2k+6=0$

Now Multiplying by 4 both Sides

$\Rightarrow 9k^2+9-18k+4k^2+36k-36-8k+24=0$

$\Rightarrow 13k^2+10k-3=0 \Rightarrow 13k^2+13k-3k-3=0$

$\Rightarrow (13k-3)(k+1)=0 \Rightarrow k=-1$ or $k=3/13$

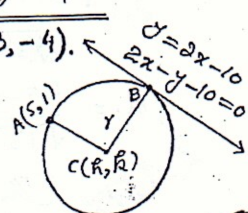
$\Rightarrow h = \dfrac{3(-1)-3}{2} = \dfrac{-6}{2} = -3$ or $h = \dfrac{3\left(\frac{3}{13}\right)-3}{2} = \dfrac{-30}{2\times 13} = \dfrac{-15}{13}$

Thus Centre is either $(-1,-3)$ or $\left(-\frac{15}{13}, \frac{3}{13}\right)$

Equations of Circle are

$$(x+1)^2+(y+3)^2=4 \quad \text{or} \quad \left(x+\tfrac{15}{13}\right)^2+\left(y-\tfrac{3}{13}\right)^2=4$$

(c) $A(5,1)$ and $2x-y-10=0$ is tangent at $B(3,-4)$.



Let $C(h,k)$ be the centre of Circle.

So $|AC| = |BC| = \text{radius} \Rightarrow |AC|^2 = |BC|^2$

$\Rightarrow (h-5)^2+(k-1)^2=(h-3)^2+(k+4)^2$

$\Rightarrow h^2+25-10h+k^2+1-2k = h^2+9-6h+k^2+16+8k$

$\Rightarrow 4h+10k=1$ —— ①

Slope of tangent $(m_1) = 2$ and Slope of $\overline{BC}\ (m_2) = \dfrac{k+4}{h-3}$

Now "radius is always perpendicular to tangent"

So $m_1 m_2 = -1 \Rightarrow 2\left[\dfrac{k+4}{h-3}\right]=-1 \Rightarrow 2k+8=3-h$

$\Rightarrow h=-5-2k$ — ②

Putting the value of h

① $\Rightarrow 4[-5-2k]+10k=1$

$\Rightarrow -20-8k+10k=1$

$\Rightarrow 2k=21 \Rightarrow \boxed{k=\dfrac{21}{2}}$

② $\Rightarrow h=-5-2\left(\frac{21}{2}\right) \Rightarrow \boxed{h=-26}$

Centre $(h,k) = \left(-26, \dfrac{21}{2}\right)$

Now $r = |BC| = \sqrt{(-26-3)^2+\left(\frac{21}{2}+4\right)^2}$

$r = \sqrt{841+\dfrac{841}{4}} = \sqrt{\dfrac{4205}{4}}$

Equation of Circle:

$$(x+26)^2+\left(y-\frac{21}{2}\right)^2=\frac{4205}{4}$$

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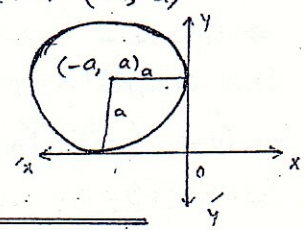
Q.5:- Radius $= a$ Circle lies in 2nd quadents with axes as tangents so Centre $(h, k) = (-a, a)$

Equation of Circle:

$(x+a)^2 + (y-a)^2 = a^2$

$x^2 + a^2 + 2ax + y^2 + a^2 - 2ay = a^2$

$\boxed{x^2 + y^2 + 2ax - 2ay + a^2 = 0}$



Q.6 $L_1: 3x - 2y = 0$ $L_2: 2x + 3y - 13 = 0$

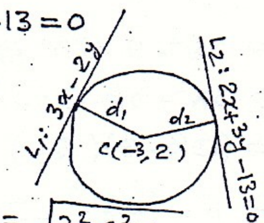
Equation of Circle: $x^2 + y^2 + 6x - 4y = 0$

Comparing with general Eq. of Circle

$2g = 6$ $2f = -4$ $c = 0$

$g = 3$ $f = -2$ $c = 0$

Centre: $C(-3, 2) = C(-g, -f)$



radius $(r) = \sqrt{f^2 + g^2 - c}$

$r = \sqrt{(3)^2 + (-2)^2 - 0} = \sqrt{9 + 4 - 0}$

$r = \sqrt{13}$

These lines will be tangents if

$d_1 = r = d_2$ (i.e. distance from centre = length of radius)

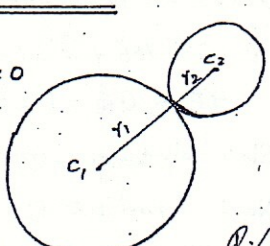
$d_1 = \dfrac{|3(-3) - 2(2)|}{\sqrt{(3)^2 + (-2)^2}}$ $d_2 = \dfrac{|2(-3) + 3(2) - 13|}{\sqrt{(2)^2 + (3)^2}}$

$d_1 = \dfrac{|-9-4|}{\sqrt{13}} = \dfrac{13}{\sqrt{13}} = \sqrt{13} = r$ $d_2 = \dfrac{|-6+6-13|}{\sqrt{13}} = \dfrac{13}{\sqrt{13}} = \sqrt{13} = r$

Thus L_1 and L_2 are tangents to given circle.

Q.7:-

$x^2 + y^2 + 2x - 2y - 7 = 0$ & $x^2 + y^2 - 6x + 4y + 9 = 0$



Comparing with General Equation of Circle

$2g = 2$, $2f = -2$, $c = -7$ | $2g = -6$, $2f = 4$, $c = 9$

$g = 1$, $f = -1$, $c = -7$ | $g = -3$, $f = 2$, $c = 9$

$C_1(-g, -f) = C_1(-1, 1)$ | $C_2(-g, -f) = C_2(3, -2)$

$r_1 = \sqrt{f^2 + g^2 - c}$ $r_2 = \sqrt{f^2 + g^2 - c}$

$r_1 = \sqrt{(1)^2 + (-1)^2 - (-7)}$ $r_2 = \sqrt{(-3)^2 + (2)^2 - (9)}$

$r_1 = \sqrt{1+1+7} = \sqrt{9} = 3$ $r_2 = \sqrt{9+4-9} = \sqrt{4} = 2$

Now $|C_1C_2| = \sqrt{(3+1)^2 + (-2-1)^2} = \sqrt{(4)^2 + (-3)^2}$

$|C_1C_2| = \sqrt{16+9} = \sqrt{25} = 5$

$|C_1C_2| = r_1 + r_2 = 5$ so Circles touch externally.

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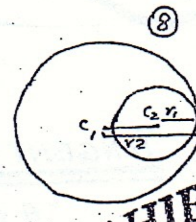
Q.8:- $x^2+y^2+2x-8=0$ & $x^2+y^2-6x+6y-46=0$



Comparing with General Equation of Circle:

$2g=2$, $2f=0$, $c=-8$

$g=1$, $f=0$, $c=-8$

Centre: $C_1(-g,-f)=C_1(-1,0)$

$r_1=\sqrt{f^2+g^2-c}$

$r_1=\sqrt{(1)^2+(0)^2-(-8)}$

$r_1=\sqrt{1+8}=\sqrt{9}=3$

$2g=-6$, $2f=6$, $c=-46$

$g=-3$, $f=3$, $c=-46$

Centre: $C_2(-g,-f)=C_2(3,-3)$

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$r_2=\sqrt{(-3)^2+(3)^2-(-46)}=\sqrt{g^2+f^2-c}$

$r_2=\sqrt{9+9+46}=\sqrt{64}=8$

$r_2=8$

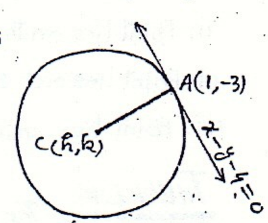
Now $|C_1C_2|=\sqrt{(3+1)^2+(-3-0)^2}=\sqrt{(4)^2+(-3)^2}=\sqrt{16+9}$

$|C_1C_2|=\sqrt{25}=5$

$\Rightarrow |C_1C_2|=r_2-r_1=5$ So Circles touch internally.

Q.9:- Radius $=2$ Tangent Line: $x-y-4=0$ at $A(1,-3)$

Let (h,k) be the centre of circle then Equation is



$$(x-h)^2+(y-k)^2=(2)^2$$

$\Rightarrow x^2+h^2-2xh+y^2+k^2-2yk=4$

Now $A(1,-3)$ Lies on Circle so must satisfy it.

$\Rightarrow 1+h^2-2h+9+k^2+6k=4$

$h^2+k^2-2h+6k+6=0$ ——— (1)

Slope of Line$(m_1)=1$

Slope of $\overline{AC}(m_2)=\frac{k+3}{h-1}$

Now radius is $\perp$ to tangent so

$m_1m_2=-1$

$\Rightarrow \left(\frac{k+3}{h-1}\right)(1)=-1 \Rightarrow k+3=1-h$

$\Rightarrow \boxed{k=-h-2}$

$\Rightarrow h^2+(-h-2)^2-2h+6(-h-2)+6=0$

$h^2+h^2+4h+4-2h-6h-12+6=0$

$2h^2-4h-2=0 \Rightarrow h^2-2h-1=0$ ——— (2)

$h=\frac{2\pm\sqrt{4+4}}{2}=1\pm\sqrt{2} \Rightarrow h=1+\sqrt{2}$ or $h=1-\sqrt{2}$

$\Rightarrow k=-1-\sqrt{2}-2$

$k=-3-\sqrt{2}$

$k=-1+\sqrt{2}-2$

$k=-3+\sqrt{2}$

Thus Centre is $C(1+\sqrt{2},-3-\sqrt{2})$ OR $C(1-\sqrt{2},-3+\sqrt{2})$

Equations are

$$\boxed{(x-1-\sqrt{2})^2+(y+3+\sqrt{2})^2=4}$$

OR

$$\boxed{(x-1+\sqrt{2})^2+(y+3-\sqrt{2})^2=4}$$

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$\Rightarrow$ Similarly Q.4, Part (d) Try yourself as Q.9.