Partial Fraction

Fraction:-

A fraction is a combination of two Polynomials, one as Denominator and other as a Neumerator.

$$\text{Fraction} = \frac{\text{Numerator}}{\text{Denominator}}$$

Rational Fraction:-

"The fraction in which the polynomials are in the form of quotient $\frac{P(x)}{Q(x)}$ is called Rational Fraction." $Q(x) \neq 0$

Rational Fraction is of two types:

Proper Rational Fraction:

"The rational fraction inwhich Power of numerator is less than Power of Denominator is called Proper rational fraction."

i.e. $\frac{4x+5}{4x^2+5x+7}$, $\frac{2}{4x+5}$ etc

Improper Rational Fraction:

The rational fraction in which Power of numerator is greater than or Equal to the power of Denominator is called Improper Rational Fraction.

i.e. $\frac{4x+7}{4x+9}$, $\frac{7x^2+9x+5}{6x+17}$ etc

Equation:-

"An open sentance which contains the Sign of Equality (=) is called Equation."

i.e. $7x+5=3$

Conditional Equation:-

"The equation which is Satisfied by some fixed value of independent variable are called Conditional Equations."

i.e. $7x+3=10$ true for $x=1$

Identity:-

The Equation which is satisfied by any value of independent variable involving in it is called Identity.

i.e. $4x^2+7x = x(4x+7)$

is Satisfied for every value of x.

Pratial Fractions:-

"The Fractions obtained after Converting a rational fraction into two or more than two fractions are called Partial Fractions."

Convertion into Partial Fractions:-

The following Steps should be used to Convert a fraction into partial fractions:

(1) The fraction should be proper (ie. Power of N. is less than Power of D)

(2) Factorize the Denominator Completely.

(3) Solve according to their nature.

Type:(1): If the Denominator Consists of linear (First degree) factors the write in the form:

$$\frac{4x+7}{(4x+3)(9x+2)} = \frac{A}{4x+3} + \frac{B}{9x+2} \quad \text{—— (Eq)}$$

(i) Multiply both sides by $(4x+3)(9x+2)$

$$4x+7 = A(9x+2) + B(4x+3) \quad \text{—— (1)}$$

(ii) Put $4x+3=0$ and $9x+2=0$ and get the Values of x

$$x = -3/4 \quad \text{and} \quad x = \frac{-2}{9}$$

and now put in Eq (1) to get (A) and B

(iii) Put A and B back in (Eq) To get Partial Fractions.

Exercise 5.1

(1) $\dfrac{1}{x^2-1} = \dfrac{1}{(x+1)(x-1)}$

$$\frac{1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1} \quad \text{—— (a)}$$

Multiply both sides by $(x+1)(x-1)$

$$1 = A(x-1) + B(x+1) \quad \text{—— (1)}$$

Put $x+1=0 \Rightarrow x=-1$

Put $x=-1$ in Eq (1)

$$1 = A(-1-1) + B(-1+1)$$

$$1 = A(-2) + 0 \Rightarrow -2A = 1$$

$$A = \frac{-1}{2}$$

Put $x-1=0 \Rightarrow x=1$

Put in Eq (1), we get

$$1 = A(1-1) + B(1+1)$$

$$1 = 0 + B2 \Rightarrow B = \frac{1}{2}$$

Putting A and B in Eq (a)

$$\frac{1}{(x+1)(x-1)} = \frac{-1/2}{(x+1)} + \frac{1/2}{(x-1)}$$

$$\frac{1}{(x+1)(x-1)} = \frac{1}{2(x-1)} - \frac{1}{2(x+1)}$$

are the required partial fractions.

(2) $\dfrac{x^2(x^2+1)}{(x+1)(x-1)}$

$= \dfrac{x^4+x^2}{x^2-1}$ $(\because (x+y)(x-y)=x^2-y^2)$

The Fraction must be Proper so

$= x^2+2+\dfrac{2}{x^2-1}$

$= x^2+2+\dfrac{2}{(x-1)(x+1)}$

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          x^2 + 2
x^2 - 1 ) x^4 + x^2
         -(x^4 - x^2)
          2x^2
         -(2x^2 - 2)
              2
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The Partilizationis

$\dfrac{2}{(x-1)(x+1)} = \dfrac{A}{x-1} + \dfrac{B}{x+1}$

Multiplying both Sides by $(x-1)(x+1)$

$2 = A(x+1) + B(x-1)$

Put $x-1=0 \Rightarrow x=1$

$2 = 2A + (0)B \Rightarrow \boxed{A=1}$

Put $x+1=0 \Rightarrow x=-1$

$2 = (0)A + (-2)B \Rightarrow \boxed{B=-1}$

$\therefore \dfrac{2}{(x-1)(x+1)} = \dfrac{1}{x-1} + \dfrac{-1}{x+1}$

Thus the Partial Fractionsare

$\dfrac{x^2(x^2+1)}{(x+1)(x-1)} = x^2+2+\dfrac{1}{x-1} - \dfrac{1}{x+1}$

(3) $\dfrac{2x+1}{(x-1)(x+2)(x+3)}$

The Partilization is

$\dfrac{2x+1}{(x-1)(x+2)(x+3)} = \dfrac{A}{x-1} + \dfrac{B}{x+2} + \dfrac{C}{x+3}$

Multiplying both Sides by $(x-1)(x+2)(x+3)$

$(2x+1) = A(x+2)(x+3) + B(x-1)(x+3) + C(x-1)(x+2)$

Put $x-1=0 \Rightarrow x=1$

$2+1 = A(3)(4) + B(0) + C(0)$

$2+1 = 12A \Rightarrow \boxed{A = \frac{3}{12} = \frac{1}{4}}$

Put $x+2=0 \Rightarrow x=-2$

$-4+1 = A(0) + B(-2-1)(1) + C(-3)(0)$

$-3 = -3B \Rightarrow \boxed{B=1}$

Put $x+3=0 \Rightarrow x=-3$

$-6+1 = (0)A + (0)B + C(-4)(-1)$

$-5 = 4C \Rightarrow \boxed{C = \frac{-5}{4}}$

$\therefore \dfrac{2x+1}{(x-1)(x+2)(x+3)} = \dfrac{1/4}{x-1} + \dfrac{1}{x+2} + \dfrac{-5/4}{x+3}$

Thus Partial fractions are

$\dfrac{2x+1}{(x-1)(x+2)(x+3)} = \dfrac{1}{4(x-1)} + \dfrac{1}{(x+2)} - \dfrac{5}{4(x+3)}$

(4) $\dfrac{3x^2-4x-5}{(x-2)(x^2+7x+10)}$

$= \dfrac{3x^2-4x-5}{(x-2)\{x^2+5x+2x+10\}}$

TAHIR

$= \dfrac{3x^2-4x-5}{(x-2)(x+2)(x+5)}$

$\dfrac{3x^2-4x-5}{(x-2)(x+2)(x+5)} = \dfrac{A}{x-2} + \dfrac{B}{x+2} + \dfrac{C}{x+5}$

Multiplying both Sides by $(x-2)(x+2)(x+5)$

$3x^2-4x-5 = A(x+2)(x+5) + B(x-2)(x+5) + C(x-2)(x+2)$

Put $x-2=0 \Rightarrow x=2$

$3(4)-4(2)-5 = A(4)(7) + (0)B + (0)C$

$-1 = 28A \implies \boxed{A = \frac{-1}{28}}$

Put $x+2=0 \implies x=-2$

$3(4)+8-5 = (0)A + B(-4)(3) + (0)C$

$15 = -12B \implies \boxed{B = \frac{-5}{4}}$

Put $x+5=0 \implies x=-5$

$3(25)-4(-5)-5 = (0)A+(0)B+C(-7)(-3)$

$90 = 21C \implies \boxed{C = \frac{30}{7}}$

Thus Partial Fractions are

$$\frac{3x^2-4x-5}{(x-2)(x^2+7x+10)} = \frac{-1/28}{x-2} + \frac{-5/4}{x+2} + \frac{30/7}{x+5}$$

$$\frac{3x^2-4x-5}{(x-2)(x^2+7x+10)} = \frac{-1}{28(x-2)} - \frac{5}{4(x+2)} + \frac{30}{7(x+5)}$$

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(5) $\dfrac{1}{(x-1)(2x-1)(3x-1)}$

$$\frac{1}{(x-1)(2x-1)(3x-1)} = \frac{A}{x-1} + \frac{B}{(2x-1)} + \frac{C}{(3x-1)}$$

Multiplying both Sides by $(x-1)(2x-1)(3x-1)$

$1 = A(2x-1)(3x-1) + B(x-1)(3x-1) + C(x-1)(2x-1)$

Put $x-1=0 \implies x=1$

$1 = A(1)(2) + (0)B + C(0)$

$1 = 2A \implies \boxed{A = \frac{1}{2}}$

Put $2x-1=0 \implies x = 1/2$

$1 = (0)A + B(\frac{-1}{2})(\frac{3}{2}-1) \Rightarrow 1 = B(\frac{-1}{2})(\frac{1}{2})$

$1 = \frac{-B}{4} \implies \boxed{B=-4}$

Put $3x-1=0 \implies x = 1/3$

$1 = (0)A + (0)B + C(\frac{1}{3}-1)(\frac{2}{3}-1)$

$1 = C(\frac{-2}{3})(\frac{-1}{3}) \implies \boxed{C = \frac{9}{2}}$

The Partial Fractions

$$\frac{1}{(x-1)(2x-1)(3x-1)} = \frac{1/2}{(x-1)} + \frac{-4}{(2x-1)} + \frac{9/2}{(3x-1)}$$

$$\frac{1}{(x-1)(2x-1)(3x-1)} = \frac{1}{2(x-1)} - \frac{4}{(2x-1)} + \frac{9}{2(3x-1)}$$

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(6) $\dfrac{x}{(x-a)(x-b)(x-c)}$

$$\frac{x}{(x-a)(x-b)(x-c)} = \frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{C}{(x-c)}$$

Multiplying both Sides by $(x-a)(x-b)(x-c)$

$x = A(x-b)(x-c) + B(x-a)(x-c) + C(x-a)(x-b)$

Put $x-a=0 \implies x=a$

$a = A(a-b)(a-c) + 0(B) + 0C$

$\boxed{A = \frac{a}{(a-b)(a-c)}}$

Put $x-b=0 \implies x=b$

$b = (0)A + B(b-a)(b-c) + 0C$

$\boxed{B = \frac{b}{(b-a)(b-c)}}$

Put $x-c=0 \implies x=c$

$c = 0A + 0B + C(c-a)(c-b)$

$\boxed{C = \frac{c}{(c-a)(c-b)}}$

Thus the Partial Fractions are

$$\frac{x}{(x-a)(x-b)(x-c)} = \frac{a}{(x-a)(a-b)(a-c)} + \frac{b}{(x-b)(b-a)(b-c)} + \frac{c}{(x-c)(c-a)(c-b)}$$

Ans.

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(7) $\dfrac{6x^3+5x^2-7}{2x^2-x-1}$

$$\begin{array}{r}
3x+4 \\
2x^2-x-1\overline{)\,6x^3+5x^2-7} \\
\underline{-6x^3\mp3x^2\mp3x} \\
8x^2+3x-7 \\
\underline{-8x^2\mp4x\mp4} \\
7x-3
\end{array}$$

$= 3x+4+\dfrac{7x-3}{2x^2-x-1}$

$= 3x+4+\dfrac{7x-3}{2x^2-2x+x-1}$

$= 3x+4+\dfrac{7x-3}{(2x+1)(x-1)}$

The partial fractions are

$\dfrac{7x-3}{(2x+1)(x-1)} = \dfrac{A}{x-1}+\dfrac{B}{2x+1}$

multiplying both sides by $(2x+1)(x-1)$

$7x-3 = A(2x+1)+B(x-1)$

Put $x-1=0 \Rightarrow x=1$

$7-3 = A(3)+0B$

$4=3A \Rightarrow \boxed{A=\frac{4}{3}}$

Put $2x+1=0 \Rightarrow x=-\frac{1}{2}$

$-\frac{7}{2}-3 = 0A+B(-\frac{1}{2}-1)$

$-\frac{13}{2} = -\frac{3}{2}B \Rightarrow \boxed{B=\frac{13}{3}}$

Thus the Partilization is

$\dfrac{6x^3+5x^2-7}{2x^2-x-1} = 3x+4+\dfrac{4}{3(x-1)}+\dfrac{13}{3(2x+1)}$

(8) $\dfrac{2x^3+x^2-5x+3}{2x^3+x^2-3x}$

$$\begin{array}{r}
1 \\
2x^3+x^2-3x\overline{)\,2x^3+x^2-5x+3} \\
\underline{-2x^3\mp x^2\pm3x} \\
-2x+3
\end{array}$$

$= 1-\dfrac{2x-3}{x(2x^2+x-3)}$

$= 1-\dfrac{2x-3}{x(2x^2+3x-2x-3)}$

$1-\dfrac{2x-3}{x(x-1)(2x+3)}$

$\dfrac{2x-3}{x(x-1)(2x+3)} = \dfrac{A}{x}+\dfrac{B}{x-1}+\dfrac{C}{2x+3}$

Multiplying both sides by $x(x-1)(2x+3)$

$2x-3 = A(x-1)(2x+3)+B(x)(2x+3)+C(x)(x-1)$

Put $x=0$

$-3 = A(-1)(3) \Rightarrow -3A=-3$

$\boxed{A=1}$

Put $x-1=0 \Rightarrow x=1$

$2-3 = (0)A+B(1)(2+3)+C(0)$

$-1 = 5B \Rightarrow \boxed{B=\frac{-1}{5}}$

Put $2x+3=0 \Rightarrow x=-3/2$

$-\frac{6}{2}-3 = (0)A+B(0)+C(-\frac{3}{2})(-\frac{3}{2}-1)$

$-6 = C(-\frac{3}{2})(-\frac{5}{2}) \Rightarrow -6 = C(\frac{15}{4})$

$C = \frac{-6\times4}{15} \Rightarrow \boxed{C=-8/5}$

Thus the Partialization is

$\dfrac{2x^3+x^2-5x+3}{2x^3+x^2-3x} = 1-\left\{\dfrac{1}{x}+\dfrac{-1/5}{(x-1)}+\dfrac{-8/5}{2x+3}\right\}$

$\dfrac{2x^3+x^2-5x+3}{2x^3+x^2-3x} = 1-\dfrac{1}{x}+\dfrac{1}{5(x-1)}+\dfrac{8}{5(2x+3)}$

Q.9 $\dfrac{(x-1)(x-3)(x-5)}{(x-2)(x-4)(x-6)}$

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$= \dfrac{(x^2-4x+3)(x-5)}{(x^2-6x+8)(x-6)}$

$= \dfrac{x^3-4x^2+3x-5x^2+20x-15}{x^3-6x^2+8x-6x^2+36x-48}$

$$= \frac{x^3-9x^2+23x-15}{x^3-12x^2+44x-48}$$

$$\begin{array}{r} 1 \\ x^3-12x^3+44x-48 \overline{\big)\, x^3-9x^2+23x-15} \\ \underline{-x^3 \overset{+}{-} 12x^2 \overset{-}{+} 44x \overset{+}{-} 48} \\ 3x^2-21x+33 \end{array}$$

$$= 1 + \frac{3x^2-21x+33}{(x-2)(x-4)(x-6)}$$

Thus Partilization is

$$\frac{3x^2-21x+33}{(x-2)(x-4)(x-6)} = \frac{A}{x-2} + \frac{B}{x-4} + \frac{C}{x-6}$$

Multiplying both Sides by $(x-2)(x-4)(x-6)$

$$3x^2-21x+33 = A(x-4)(x-6)+B(x-2)(x-6)+C(x-2)(x-4)$$

Put $x-2=0 \Rightarrow x=2$

$$3(4)-21(2)+33 = A(-2)(-4)+B(0)+C(0)$$

$$12+33-42 = 8A \Rightarrow \boxed{A=\frac{3}{8}}$$

Put $x-4=0 \Rightarrow x=4$

$$3(16)-21(4)+33 = (0)A+B(2)(-2)+(0)C$$

$$48-84+33 = -4B \Rightarrow -3=-4B$$

$$\boxed{B=\frac{3}{4}}$$

TAHIR

Put $x-6=0 \Rightarrow x=6$

$$3(36)-21(6)+33 = (0)A+(0)B+C(4)(2)$$

$$108-126+33 = 8C \Rightarrow \boxed{C=\frac{15}{8}}$$

Thus the Partilization is

$$\frac{(x-1)(x-3)(x-5)}{(x-2)(x-4)(x-5)} = 1 + \frac{3/8}{x-2} + \frac{3/4}{x-4} + \frac{15/8}{x-6}$$

$$\frac{(x-1)(x-3)(x-5)}{(x-2)(x-4)(x-6)} = 1 + \frac{3}{8(x-2)} + \frac{3}{4(x-4)} + \frac{15}{8(x-6)}$$

Q.10 $\dfrac{1}{(1-ax)(1-bx)(1-cx)}$

$$\frac{1}{(1-ax)(1-bx)(1-cx)} = \frac{A}{(1-ax)} + \frac{B}{(1-bx)} + \frac{C}{(1-cx)}$$

Multiplying both Sides by $(1-ax)(1-bx)(1-cx)$

$$1 = A(1-bx)(1-cx)+B(1-ax)(1-cx)+C(1-ax)(1-bx)$$

Put $1-ax=0 \Rightarrow x=\frac{1}{a}$

$$1 = A\left(1-\frac{b}{a}\right)\left(1-\frac{c}{a}\right)+0B+0C \Rightarrow 1 = A\left(\frac{a-b}{a}\right)\left(\frac{a-c}{a}\right)$$

$$\boxed{A=\frac{a^2}{(a-b)(a-c)}},$$

Put $(1-bx)=0 \Rightarrow x=1/b$

$$1 = (0)A+B\left(1-\frac{a}{b}\right)\left(1-\frac{c}{b}\right)+0C$$

$$1 = B\left(\frac{b-a}{b}\right)\left(\frac{b-c}{b}\right) \Rightarrow \boxed{B=\frac{b^2}{(b-a)(b-c)}}$$

Put $(1-cx)=0 \Rightarrow x=1/c$

$$1 = (0)A+(0)B+C\left(1-\frac{b}{c}\right)\left(1-\frac{a}{c}\right)$$

$$1 = C\left(\frac{c-b}{c}\right)\left(\frac{c-a}{c}\right) \Rightarrow \boxed{C=\frac{c^2}{(c-b)(c-a)}}$$

Thus The Partilization is

$$\frac{1}{(1-ax)(1-bx)(1-cx)} = \frac{a^2}{(1-ax)(a-b)(a-c)} + \frac{b^2}{(1-bx)(b-a)(b-c)} + \frac{c^2}{(1-cx)(c-b)(c-a)}$$

Q.11 $\dfrac{x^2+a^2}{(x^2+b^2)(x^2+c^2)(x^2+d^2)}$

Let $x^2=y$ then

$$\frac{y+a^2}{(y+b^2)(y+c^2)(y+d^2)}$$

$$\frac{y+a^2}{(y+b^2)(y+c^2)(y+d^2)} = \frac{A}{y+b^2} + \frac{B}{y+c^2} + \frac{C}{y+d^2}$$

Multiplying both Sides by $(y+b^2)(y+c^2)(y+d^2)$

$$y+a^2 = A(y+c^2)(y+d^2) + B(y+b^2)(y+d^2) + C(y+b^2)(y+c^2)$$

Put $y+b^2=0 \Longrightarrow y=-b^2$

$$a^2-b^2 = A(c^2-b^2)(d^2-b^2) + (0)B + (0)C$$

$$\boxed{A = \frac{a^2-b^2}{(c^2-b^2)(d^2-b^2)}}$$

Put $y+c^2=0 \Longrightarrow y=-c^2$

$$a^2-c^2 = (0)A + B(b^2-c^2)(d^2-c^2) + C(0)$$

$$\boxed{B = \frac{a^2-c^2}{(b^2-c^2)(d^2-c^2)}}$$

Put $y+d^2=0 \Longrightarrow y=-d^2$

$$a^2-d^2 = (0)A + (0)B + C(b^2-d^2)(c^2-d^2)$$

$$\boxed{C = \frac{a^2-d^2}{(b^2-d^2)(c^2-d^2)}}$$

Thus Partilization is

$$\frac{y+a^2}{(y+b^2)(y+c^2)(y+d^2)} = \frac{a^2-b^2}{(y+b^2)(c^2-b^2)(d^2-b^2)} + \frac{a^2-c^2}{(y+c^2)(b^2-c^2)(d^2-c^2)} + \frac{a^2-d^2}{(y+d^2)(b^2-d^2)(c^2-d^2)}$$

Now put back $y=x^2$

$$\frac{x^2+a^2}{(x^2+b^2)(x^2+c^2)(x^2+d^2)} = \frac{a^2-b^2}{(x^2+b^2)(c^2-b^2)(d^2-b^2)} + \frac{a^2-c^2}{(x^2+c^2)(b^2-c^2)(d^2-c^2)} + \frac{a^2-d^2}{(x^2+d^2)(b^2-d^2)(c^2-d^2)}$$

Type-II: In Case of repeated linear factor, write upto n powers.

$$\frac{x}{(1+x)^3} = \frac{A}{1+x} + \frac{B}{(1+x)^2} + \frac{C}{(1+x)^3}$$

Exercise 5.2

Q.1 $\dfrac{2x^2-3x+4}{(x-1)^3}$

$$\frac{2x^2-3x+4}{(x-1)^3} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3}$$

Multiplying both Sides by $(x-1)^3$

$$2x^2-3x+4 = A(x-1)^2 + B(x-1) + C$$

Put $x-1=0 \Longrightarrow x=1$

$$2-3+4 = (0)A + 0(B) + C$$

$$\boxed{C=3}$$

Comparing the Coefficients of

$x^2 \Longrightarrow \boxed{2=A}$

$x \Longrightarrow -3 = -2A + B$

$$B = -3+2A$$

$$B = -3+2(2) = -3+4$$

$$\boxed{B=1}$$

Thus Partilization is

$$\frac{2x^2-3x+4}{(x-1)^3} = \frac{2}{x-1} + \frac{1}{(x-1)^2} + \frac{3}{(x-1)^3}$$

Q.2 $\dfrac{5x^2-2x+3}{(x+2)^2}$

$$= \frac{5x^2-2x+3}{x^2+4x+4}$$

$$\begin{array}{r|l} & \quad 5 \\ x^2+4x+4 & \overline{5x^2-2x+3} \\ & \underline{5x^2+20x+20} \\ & -22x-17 \end{array}$$

$$= 5 - \frac{22x+17}{x^2+4x+4}$$

$$= 5 - \frac{22x+17}{(x+2)^2}$$