Decreasing Function:-

The function $f(x)$ is said to be decreasing function if $f'(x) < 0$.

Stationary Point:-

"The point where function is neither increasing nor decreasing is called Stationary Point."

At stationary point $f'(x) = 0$

* Stationary Point is also called Point of inflexion if $f(x)$ is neither increasing nor decreasing.

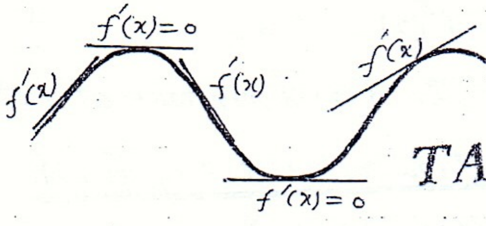
Second Derivative Test:-

Let f is a differentiable function at $x = a$ then

(i) f is said to be maximum at $x = a$ if $f''(x) < 0$

(ii) f is said to be minimum at $x = a$ if $f''(x) > 0$

* To get the extreme values, we put $\frac{dy}{dx}$ or $f'(x) = 0$ called Critical Points.
* The values of x for which f is either maximimum or minimum are called extreme values.



Exercise 2.9

Q.1 Determine the increasing and decreasing interval of f if

(i) $f(x) = \sin x \qquad x \in [-\pi, \pi]$

$f'(x) = \cos x$

x	$-\pi$	$-\frac{5\pi}{6}$	$-\frac{2\pi}{3}$	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{6}$	0
$f'(x)$	-1	-0.87	-0.5	0	0.5	0.87	1
x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\pi/2$	$2\pi/3$	$5\pi/6$	π
$f'(x)$	1	0.87	0.5	0	-0.5	-0.87	-1

From the table

f is increasing for $]-\frac{\pi}{2}, \frac{\pi}{2}[$

f is decreasing for $[-\pi, -\frac{\pi}{2}[$, $]\frac{\pi}{2}, \pi]$

(ii) $f(x) = \cos x \qquad x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$f'(x) = -\sin x$

x	$-\pi/2$	$-\pi/3$	$-\pi/6$	0	$\pi/6$	$\pi/3$	$\pi/2$
$f'(x)$	1	0.87	0.5	0	-0.5	-0.87	-1

From the table

f is increasing for $[-\frac{\pi}{2}, 0[$

f is decreasing for $]0, -\pi/2]$

(iii) $f(x) = 4 - x^2 \qquad x \in [-2, 2]$

$f'(x) = 0 - 2x$

$f'(x) = -2x$

x	-2	-1	0	1	2
$f'(x)$	4	2	0	-2	-4

From the table

f is increasing for $[-2,0[$

f is decreasing for $]0,2]$

(iv) $f(x)=x^2+3x+2 \qquad x\in[-4,1]$

$f'(x)=2x+3$

x	-4	-3.5	-3	-2.5	-2
$f'(x)$	-5	-4	-3	-2	-1
x	-1.5	-1	-0.5	0	1
$f'(x)$	0	1	2	3	5

Clearly from the table

f is increasing for $]-1.5,1]$

f is decreasing for $[-4,-1.5[$

Q.2 Find the extreme values for the functions defined as:

(i) $f(x)=1-x^3$

$f'(x)=0-3x^2=-3x^2$

For extreme values $f'(x)=0$

$\Rightarrow \quad -3x^2=0 \Rightarrow x^2=0 \qquad -3\neq 0$

$\Rightarrow \quad x=0$

$f''(x)=-6x$ TAHIR

$f''(x)=0 \quad$ for $x=0$

$y=1-x^3 \Rightarrow y=1 \qquad \because y=f(x)$

Thus Pt $(0,1)$ is point of inflexion.

(ii) $f(x)=x^2-x-2$

$f'(x)=2x-1$

For extreme values $f'(x)=0$

$2x-1=0 \Rightarrow x=1/2$

$f''(x)=2>0$

Thus $f(x)$ is minimum at $x=1/2$

Minimum Value of function:

$f(1/2)=(\frac{1}{2})^2-(1/2)-2=\frac{1}{4}-\frac{1}{2}-2$

$f(1/2)=\frac{1-2-8}{4}=-9/4$ Ans.

(iii) $f(x)=5x^2-6x+2$

$f'(x)=10x-6$ TAHIR

For extreme values $f'(x)=0$

$10x-6=0\Rightarrow x=\frac{3}{5}$

$f''(x)=10>0$

Thus $f(x)$ is minimum at $x=3/5$

Minimum Value of function:-

$f(3/5)=5(3/5)^2-6(3/5)+2$

$f(3/5)=5(\frac{9}{25})-\frac{18}{5}+2=\frac{9}{5}-\frac{18}{5}+2$

$f(3/5)=\frac{9-18+10}{5}=\frac{19-18}{5}=\frac{1}{5}$

$f(3/5)=1/5$ Ans.

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(iv) $f(x) = 3x^2$

$f'(x) = 6x$

For extreme values $f'(x) = 0$

$\Rightarrow 6x = 0 \Rightarrow x = 0 \qquad 6 \neq 0$

$f''(x) = 6 > 0$

Thus $f(x)$ is minimum at $x = 0$

<u>Minimum value of function:-</u>

$f(0) = 3(0)^2 = 0$ <u>Ans</u>.

(v) $f(x) = 3x^2 - 4x + 5$

$f'(x) = 6x - 4$

For extreme values $f'(x) = 0$

$\Rightarrow 6x - 4 = 0 \Rightarrow x = 2/3$

$f''(x) = 6 > 0$

Thus $f(x)$ is minimum at $x = 2/3$

<u>Minimum Value of function:-</u>

$f\left(\frac{2}{3}\right) = 3\left(\frac{2}{3}\right)^2 - 4\left(\frac{2}{3}\right) + 5 = 3\left(\frac{4}{9}\right) - \frac{8}{3} + 5$

$= \frac{4}{3} - \frac{8}{3} + 5 = \frac{4 - 8 + 15}{3} = \frac{11}{3}$

$f(2/3) = 11/3$ <u>Ans</u>.

(vi) $f(x) = 2x^3 - 2x^2 - 36x + 3$

$f'(x) = 6x^2 - 4x - 36$

For extreme values $f'(x) = 0$

$\Rightarrow 6x^2 - 4x - 36 = 0 \Rightarrow 3x^2 - 2x - 18 = 0$

$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(-18)}}{2(3)} = \frac{2 \pm \sqrt{4 + 216}}{6}$

$x = \frac{2 \pm \sqrt{220}}{6} = \frac{2 \pm 2\sqrt{55}}{6}$

$x = \frac{1 \pm \sqrt{55}}{3} \Rightarrow x = \frac{1 + \sqrt{55}}{3}, \frac{1 - \sqrt{55}}{3}$

$f''(x) = 12x - 4 = 4[3x - 1]$

$f''(x) = 4\left[3\left(\frac{1+\sqrt{55}}{3}\right) - 1\right] = 4\sqrt{55} > 0$

So $f(x)$ is minimum at $x = \frac{1 + \sqrt{55}}{3}$

<u>Minimum Value of function:-</u>

$f\left[\frac{1+\sqrt{55}}{3}\right] = \frac{-1}{27}\left(247 + 220\sqrt{55}\right)$ <u>Ans</u>

$\Rightarrow f''(x) = 4[3x - 1]$

$f''(x) = 4\left[3\left(\frac{1-\sqrt{55}}{3}\right) - 1\right] = -4\sqrt{55} < 0$

Thus $f(x)$ is maximum at $x = \frac{1 - \sqrt{55}}{3}$

<u>Maximum Value of function:-</u>

$f\left[\frac{1-\sqrt{55}}{3}\right] = \frac{1}{27}\left(-247 + 220\sqrt{55}\right)$ <u>Ans</u>

(vii) $f(x) = x^4 - 4x^3$

$f'(x) = 4x^3 - 12x^2$

For extreme values $f'(x) = 0$

$\Rightarrow 4x^3 - 12x^2 = 0 \Rightarrow x^2(4x - 12) = 0$

$x = 0 \;,\; x = 12/4 \Rightarrow x = 3$

$x = 0, 3$

$f''(x) = 12x^2 - 24x$

$f''(x) = 12(0)^2 - 24(0) = 0 \quad$ for $x = 0$

$y = (0)^4 - 4(0)^3 = 0 \qquad \because y = f(x)$

Thus $(0, 0)$ is point of inflexion.

$f''(x) = 12(3)^2 - 24(3) = 12(9) - 24(3)$

$f''(x) = 108 - 72 = 36 > 0$

Thus $f(x)$ is minimum at $x = 3$

Minimum value of function.

$f(3) = (3)^4 - 4(3)^3 = 81 - 108 = -27$ Ans.

(viii) $f(x) = (x-2)^2(x-1)$

$f'(x) = 2(x-2)\cdot 1\cdot(x-1) + (x-2)^2\cdot(1-0)$

$f'(x) = 2(x-2)(x-1) + (x-2)^2$

$f'(x) = 2(x^2-3x+2) + (x^2+4-4x)$

$f'(x) = 2x^2-6x+4+x^2-4x+4$

$f'(x) = 3x^2-10x+8$

For extreme values $f'(x)=0$

$\Rightarrow 3x^2-10x+8=0$

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$3x^2-6x-4x+8=0$

$3x(x-2)-4(x-2)=0$

$x-2=0 \quad \wedge \quad 3x-4=0$

$x=2 \quad \wedge \quad x=4/3$

$f''(x) = 6x-10$

$f''(x) = 6(2)-10 = 12-10 = 2>0$

So $f(x)$ is minimum at $x=2$

Minimum value of function.

$f(2) = (2-2)^2\cdot(x-1) = 0$ Ans.

Now $f''(x) = 6(\frac{4}{3})-10 = 8-10 = -2<0$

So $f(x)$ is maximum at $x=4/3$

Maximum value of function:-

$f(4/3) = (\frac{4}{3}-2)^2(\frac{4}{3}-1) = (\frac{-2}{3})^2(\frac{1}{3})$

$f(4/3) = \frac{4}{9}\cdot\frac{1}{3} = \frac{4}{27}$ Ans.

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(ix) $f(x) = 5+3x-x^3$

$f'(x) = 0+3\cdot 1-3x^2 = 3-3x^2$

For extreme values $f'(x)=0$

$\Rightarrow 3-3x^2=0 \Rightarrow 1-x^2=0 \quad 3\neq 0$

$\Rightarrow x^2=1 \Rightarrow x=\pm 1$

$f''(x) = 0-6x = -6x$

$f''(x) = -6(1) = -6<0$ at $x=1$

Thus $f(x)$ is maximum at $x=1$

$f''(x) = -6(-1) = 6>0$ at $x=-1$

Thus $f(x)$ is minimum at $x=-1$

Max.Value $=7$ Min.Value $=3$

Q.3 $f(x) = \sin x+\cos x \quad x\in[0,2\pi]$

$f'(x) = \cos x-\sin x$

For extreme values $f'(x)=0$

$\Rightarrow \cos x-\sin x=0$

$\cos x = \sin x$

This is possible if $x=\frac{\pi}{4}, \frac{5\pi}{4} \in[0,2\pi]$

$f''(x) = -\sin x-\cos x$

$f''(x) = -\sin\pi/4-\cos\pi/4$ at $x=\pi/4$

$f''(x) = -\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}} = \frac{-2}{\sqrt{2}} = -\sqrt{2}<0$

$f(x)$ is maximum at $x=\pi/4$

Maximum Value of function.

$f(\pi/4) = \sin\pi/4+\cos\pi/4 = \frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}$

$f(\pi/4) = \frac{2}{\sqrt{2}} = \sqrt{2}$ Ans.

$\Rightarrow f''(x) = -\sin\frac{5\pi}{4}-\cos\frac{5\pi}{4}$

$f''(x) = -(\frac{-1}{\sqrt{2}})-(\frac{-1}{\sqrt{2}}) = \frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}}$

$f''(x) = \sqrt{2}>0$

$f''(x) > 0 \quad$ at $\quad x = \frac{5\pi}{4}$

Thus $f(x)$ is minimum at $x = \frac{5\pi}{4}$

Minimum Value of function.

$$f\left(\frac{5\pi}{4}\right) = \sin\left(\frac{5\pi}{4}\right) + \cos\left(\frac{5\pi}{4}\right)$$

$$= -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = \frac{-2}{\sqrt{2}} = -\sqrt{2} \text{ Ans}$$

Q.4 Show $y = \frac{\ln x}{x}$ is maximum at $x = e$

Proof:- $\quad y = \frac{\ln x}{x}$

$$\frac{dy}{dx} = \frac{x(1/x) - (\ln x)\cdot 1}{x^2}$$

$$\frac{dy}{dx} = \frac{(1 - \ln x)}{x^2} \Rightarrow \frac{dy}{dx} = 0 \text{ at } x = e$$

Diff. again w.r.t. "x"

$$\frac{d^2y}{dx^2} = \frac{x^2(0 - 1/x) - (1 - \ln x)\cdot 2x}{x^4}$$

$$\frac{d^2y}{dx^2} = \frac{-x - 2x + 2x\ln x}{x^4}$$

$$d^2y/dx^2 = \frac{-1 - 2 + 2\ln x}{x^3}$$

$$d^2y/dx^2 = \frac{-3 + 2\ln x}{x^3}$$

$$d^2y/dx^2 \text{ at } x = e = \frac{-3 + 2\ln e}{e^3}$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=e} = \frac{-3+2}{e^3} = \frac{-1}{e^3} < 0$$

(Proved).

Thus $y = \frac{\ln x}{x}$ is maximum at $x = e$

Q.5 Show $y = x^x$ is minimum at $x = \frac{1}{e}$

$$y = x^x \Rightarrow \ln y = \ln x^x$$

$$\ln y = x \cdot \ln x$$

Diff. w.r.t "x"

$$\frac{1}{y}\frac{dy}{dx} = 1(\ln x) + x \cdot 1/x$$

$$\frac{dy}{dx} = y(1 + \ln x) \Rightarrow \frac{dy}{dx} = 0 \text{ at } x = 1/e$$

$$\frac{d^2y}{dx^2} = \frac{dy}{dx}(1 + \ln x) + y(0 + 1/x)$$

$$d^2y/dx^2 = \left(y(1 + \ln x)\right)(1 + \ln x) + y/x$$

$$d^2y/dx^2 = y(1 + \ln x)^2 + y/x$$

$$d^2y/dx^2 = y\left[\frac{1}{x} + (1 + \ln x)^2\right]$$

$$\frac{d^2y}{dx^2} = x^x\left[\frac{1}{x} + (1 + \ln x)^2\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{at\ x=1/e} = e^{1/x}\left[\frac{1}{(1/e)} + \left(1 + \ln\frac{1}{e}\right)^2\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=1/e} = \left(\frac{1}{e}\right)^{1/e}\left[e + (1 - 1)^2\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=1/e} = (1/e)^{1/e}\left[e + 0\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=1/e} = \left(\frac{1}{e}\right)^{1/e} \cdot e > 0$$

(Proved)

Thus $y = x^x$ is minimum at $x = 1/e$

Some Useful Formulas

*Area of Rectangle (A) = length × Width

*Perimeter of Ractangle = 2(length + Width)

*Volume of Cube = length × Width × height

*Area of triangle $\Delta = \sqrt{S(S-a)(S-b)(S-c)}$

where $S = \frac{a+b+c}{2} = \frac{P}{2}$

where P is perimeter of Δ

$P = a + b + c$