Diff. again w.r.t. x

$$(1-x^2)2y_1y_2 - 2xy_1^2 = 4y_1$$

$$2y_1\left[(1-x^2)y_2 - xy_1\right] = 4y_1$$

$$(1-x^2)y_2 - xy_1 = 2$$

$$(1-x^2)y_2 - xy_1 - 2 = 0 \text{ (Proved)}$$

Q.9 If $y = a\cos(\ln x) + b\sin(\ln x)$

Prove that $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

Proof:: $y = a\cos(\ln x) + b\sin(\ln x)$

Diff. w.r.t. x

$$\frac{dy}{dx} = a\left(-\frac{\sin(\ln x)}{x}\right) + b\frac{\cos(\ln x)}{x}$$

$$\frac{dy}{dx} = \frac{-a\sin(\ln x) + b\cos(\ln x)}{x}$$

$$x\frac{dy}{dx} = -a\sin(\ln x) + b\cos(\ln x)$$

Diff. again w.r.t "x"

$$x\frac{d^2y}{dx^2} + 1.\frac{dy}{dx} = -\frac{a\cos(\ln x)}{x} - \frac{b\sin(\ln x)}{x}$$

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{-a\cos(\ln x) - b\sin(\ln x)}{x}$$

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -(a\cos(\ln x) + b\sin(\ln x))$$

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -y$$

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0 \text{ (Proved).}$$

Maclaurin Series:-

If $f(x)$ is a real valued and Continous function then

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots$$

is called the Maclaurin's Series. This Series was introduced by Maclaurins in 1742.

Taylor's Series:-

If f is a real valued and Continuous function then

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots$$

is called Taylor's Series of $f(x+h)$. This Series was introduced by a British Mathematician Taylor.

Exercise 2.8

Q.1 Apply Maclaurins Series to prove that:

(i) $\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

Let $f(x) = \ln(x+1)$

$$f(x) = \ln(x+1) \Rightarrow f(0) = \ln(1+0) = 0$$

$$f'(x) = \frac{1}{x+1} = (x+1)^{-1} \Rightarrow f'(0) = (0+1)^{-1} = 1$$

$$f''(x) = -1(x+1)^{-2} \Rightarrow f''(0) = -1$$

$$f'''(x) = 2(x+1)^{-3} \Rightarrow f'''(0) = 2$$

$\vdots \qquad \vdots$

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Using Maclaurin's Series

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

$$\ln(x+1) = 0 + x(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(2) + \cdots$$

$$\ln(x+1) = x - \frac{x^2}{2!} + \frac{2x^3}{3!} - \cdots$$

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{2x^3}{6} - \cdots$$

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad \text{(Proved)}$$

(ii) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

Let $f(x) = \cos x$

$f(x) = \cos x \Rightarrow f(0) = \cos 0 = 1$

$f'(x) = -\sin x \Rightarrow f'(0) = 0$

$f''(x) = -\cos x \Rightarrow f''(0) = -1$

$f'''(x) = \sin x \Rightarrow f'''(0) = 0$

$f^{iv}(x) = \cos x \Rightarrow f^{iv}(0) = 1$

$\vdots \qquad \vdots$

Using Maclaurin's Series.

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

$$\cos x = 1 + x(0) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(1) + \cdots$$

$$\cos x = 1 + 0 - \frac{x^2}{2!} + 0 + \frac{x^4}{4!} + 0 + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad \text{(Proved)}$$

(iii) $\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + \cdots$

Let $f(x) = \sqrt{1+x}$

$f(x) = (1+x)^{1/2} \Rightarrow f(0) = 1$

$f'(x) = \frac{1}{2}(1+x)^{-1/2} \Rightarrow f'(0) = \frac{1}{2}$

$f''(x) = \frac{-1}{4}(1+x)^{-3/2} \Rightarrow f''(0) = \frac{-1}{4}$

$f'''(x) = \frac{3}{8}(1+x)^{-5/2} \Rightarrow f'''(0) = \frac{3}{8}$

$\vdots \qquad \vdots$

Using Maclaurins Series.

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

$$\sqrt{1+x} = 1 + x\left(\frac{1}{2}\right) + \frac{x^2}{2!}\left(\frac{-1}{4}\right) + \frac{x^3}{3!}\left(\frac{3}{8}\right) + \cdots$$

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{3x^3}{6 \cdot 8} + \cdots$$

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} + \cdots \quad \text{(Proved)}$$

(iv) $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots$

Let $f(x) = e^x$

$f(x) = e^x \Rightarrow f(0) = e^0 = 1$

$f'(x) = e^x \Rightarrow f'(0) = e^0 = 1$

$f''(x) = e^x \cdot 1 \Rightarrow f''(0) = e^0 = 1$

$f'''(x) = e^x \Rightarrow f'''(0) = e^0 = 1$

$\vdots \qquad \vdots$

Using Maclaurin's Series

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

$$e^x = 1 + x(1) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(1) + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad \text{(Proved)}$$

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(v) $e^{2x} = 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \cdots$

Let $f(x) = e^{2x}$

$f(x) = e^{2x} \Rightarrow f(0) = e^0 = 1$

$f'(x) = 2e^{2x} \Rightarrow f'(0) = 2e^0 = 2$

$f''(x) = 4e^{2x} \Rightarrow f''(0) = 4e^0 = 4$

$f'''(x) = 8e^{2x} \Rightarrow f'''(0) = 8e^0 = 8$

$\vdots \qquad \vdots$

Using Maclaurin's Series

$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots$

$e^{2x} = 1 + x(2) + \frac{x^2}{2!}(4) + \frac{x^3}{3!}(8) + \cdots$

$e^{2x} = 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \cdots$ (Proved)

Q.2 Show that is:

$\cos(x+h) = \cos x - h\sin x - \frac{h^2}{2!}\cos x + \cdots$

and evaluate $\cos 61^\circ$?

Proof:- Let $f(x+h) = \cos(x+h)$

$f(x) = \cos x$

$f'(x) = -\sin x$

$f''(x) = -\cos x$

$f'''(x) = \sin x$

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$\vdots \qquad \vdots$

Using Taylor's Series

$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots$

$\cos(x+h) = \cos x + h(-\sin x) + \frac{h^2}{2!}(-\cos x) + \frac{h^3}{3!}(\sin x) + \cdots$

$\cos(x+h) = \cos x - h\sin x - \frac{h^2}{2!}\cos x + \frac{h^3}{3!}\sin x + \cdots$ (Proved)

To evaluate $\cos 61^\circ$

Let $x + h = 61^\circ$ and $x = 60^\circ$

$h = 61^\circ - 60^\circ = 1^\circ = 0.0174$

$\cos 61^\circ = \cos 60^\circ - (0.0174)\sin 60^\circ - \frac{(0.0174)^2}{2!}\cos 60^\circ + \frac{(0.0174)^3}{3!}\sin 60^\circ + \cdots$

$\cos 61^\circ = 0.5 - 0.0151 - 0.00001569 + \cdots$

$\cos 61^\circ = 0.4848$ Ans.

Q.3 Show that

$2^{x+h} = 2^x\left[1 + (\ln 2)h + \frac{h^2}{2!}(\ln 2)^2 + \frac{h^3}{3!}(\ln 2)^3 + \cdots\right]$

Proof:- Let $f(x+h) = 2^{x+h}$

$f(x) = 2^x$

$f'(x) = 2^x \cdot \ln 2$

$f''(x) = 2^x \cdot (\ln 2)^2$

$f'''(x) = 2^x (\ln 2)^3$

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$\vdots \qquad \vdots$

Using Taylor's Theorem (Series)

$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots$

$2^{x+h} = 2^x + h2^x(\ln 2) + \frac{h^2}{2!}2^x(\ln 2)^2 + \frac{h^3}{3!}2^x(\ln 2)^3 + \cdots$

$2^{x+h} = 2^x\left[1 + h(\ln 2) + \frac{h^2}{2!}(\ln 2)^2 + \frac{h^3}{3!}(\ln 2)^3 + \cdots\right]$

(Proved)

Increasing Function:-

The function $f(x)$ is said to be increasing if

$f'(x) > 0$.

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