Exercise 2.7

Q.1 Find y_2 if:

(i) $y = 2x^5 - 3x^4 + 4x^3 + x - 2$

Diff. w.r.t. "x"

$y_1 = 2(5x^4) - 3(4x^3 \cdot 1) + 4(3x^2 \cdot 1) + 1 - 0$

$y_1 = 10x^4 - 12x^3 + 12x^2 + 1$

Diff again w.r.t. x

$y_2 = 10(4x^3) - 12(3x^2) + 12(2x) + 0$

$y_2 = 40x^3 - 36x^2 + 24x$ Ans.

(ii) $y = (2x+5)^{3/2}$

Diff. w.r.t. "x"

$y_1 = \frac{3}{2}(2x+5)^{1/2}\,(2 \cdot 1 + 0)$

$y_1 = 3(2x+5)^{1/2}$

Diff. w.r.t. "x" again

$y_2 = 3 \cdot \frac{1}{2}(2x+5)^{-1/2} \cdot (2 \cdot 1 + 0)$

$y_2 = 3(2x+5)^{-1/2} = \frac{3}{\sqrt{2x+5}}$ Ans.

(iii) $y = \sqrt{x} + \frac{1}{\sqrt{x}}$

$y = x^{1/2} + x^{-1/2}$

Diff. w.r.t. "x"

$y_1 = \frac{1}{2}x^{-1/2} \cdot 1 - \frac{1}{2} \cdot x^{-3/2} \cdot 1$

$y_1 = \frac{x^{-1/2}}{2} - \frac{x^{-3/2}}{2}$

Diff. again w.r.t "x"

$y_2 = \frac{1}{2}\left(\frac{-1}{2}x^{-3/2}\right) - \frac{1}{2}\left(\frac{-3}{2}x^{-5/2}\right)$

$y_2 = -\frac{1}{4}x^{-3/2} + \frac{3}{4}x^{-5/2}$

$y_2 = \frac{3}{4}x^{-5/2} - \frac{1}{4}x^{-3/2}$

$y_2 = \frac{1}{4}x^{-3/2}\left[3x^{-1} - 1\right]$

$y_2 = \frac{1}{4x^{3/2}}\left(\frac{3}{x} - 1\right)$

$y_2 = \frac{1}{4x^{3/2}}\left(\frac{3-x}{x}\right) = \frac{3-x}{4x^{5/2}}$ Ans.

Q.2 Find y_2 if:

(i) $y = x^2 e^{-x}$

Diff. w.r.t. "x"

$y_1 = x^2(-e^{-x}) + e^{-x}(2x \cdot 1)$

$y_1 = 2xe^{-x} - x^2e^{-x} = e^{-x}(2x - x^2)$

Diff. again w.r.t. "x"

$y_2 = e^{-x}(2 - 2x) + (2x - x^2)(e^{-x} \cdot -1)$

$y_2 = 2e^{-x} - 2xe^{-x} - 2e^{-x}x + x^2e^{-x}$

$y_2 = 2e^{-x} - 4xe^{-x} + x^2e^{-x}$

$y_2 = e^{-x}(2 - 4x + x^2)$ Ans.

(ii) $y = \ln\left(\frac{2x+3}{3x+2}\right)$

$y = \ln(2x+3) - \ln(3x+2)$

Diff. w.r.t. "x"

$y_1 = \frac{1}{2x+3} \cdot 2 - \frac{1}{3x+2} \cdot 3$

$y_1 = \frac{2}{(2x+3)} - \frac{3}{(3x+2)}$

$y_1 = 2(2x+3)^{-1} - 3(3x+2)^{-1}$

Diff. w.r.t. "x" again

$y_2 = 2\left[-1(2x+3)^{-2}\right]2 - 3\left[-1(3x+2)^{-2}\right] \cdot 3$

TAHIR

$$y_2 = -4(2x+3)^{-2} + 9(3x+2)^{-2}$$

$$y_2 = \frac{-4}{(2x+3)^2} + \frac{9}{(3x+2)^2} \quad \text{Ans.}$$

(iii) $y = \sqrt{\frac{1-x}{1+x}}$

$$y = \left(\frac{1-x}{1+x}\right)^{1/2}$$

Diff. w.r.t "x"

$$y_1 = \frac{1}{2}\left(\frac{1-x}{1+x}\right)^{-1/2} \cdot \frac{(1+x)(0-1)-(1-x)(0+1)}{(1+x)^2}$$

$$y_1 = \frac{1}{2}\left(\frac{1+x}{1-x}\right)^{1/2} \cdot \frac{-\cancel{x}-1+\cancel{x}}{(1+x)^2}$$

$$y_1 = \frac{1}{\cancel{2}}\frac{(1+x)^{1/2}}{(1-x)^{1/2}} \cdot \frac{-\cancel{2}}{(1+x)^2} = \frac{1}{(1-x)^{1/2} \cdot (1+x)^{3/2}}$$

$$y_1 = \frac{1}{(1-x)^{1/2} \cdot (1+x)^{3/2}}$$

Diff. w.r.t "x" again

$$y_2 = \frac{(1-x)^{1/2} \cdot (1+x)^{3/2} \cdot (0) - 1 \cdot \frac{d}{dx}\left[(1-x)^{1/2} \cdot (1+x)^{3/2}\right]}{\left[(1-x)^{1/2} \cdot (1+x)^{3/2}\right]^2}$$

$$y_2 = \frac{0 - \frac{d}{dx}\left[(1-x)^{1/2} \cdot (1+x)^{3/2}\right]}{(1-x)(1+x)^3}$$

$$y_2 = \frac{-\left[\frac{1}{2}(1-x)^{-1/2} \cdot (1+x)^{3/2} + \frac{3}{2}(1+x)^{1/2} \cdot (1-x)^{1/2}\right]}{(1-x)(1+x)^3}$$

$$y_2 = \frac{-\left[\frac{1}{2}\frac{(1+x)^{3/2}}{(1-x)^{1/2}} + \frac{3}{2}\left[(1-x)(1+x)\right]^{1/2}\right]}{(1-x)(1+x)^3}$$

$$y_2 = \frac{-\frac{1}{2}\frac{(1+x)^{3/2}}{\sqrt{1-x}} + \frac{3}{2}\sqrt{1-x^2}}{(1+x)^3 \cdot (1-x)} \quad \text{Ans.}$$

(32)

Q.3 Find y_2 if:

(i) $y^2 + x^2 = a^2$

Diff. w.r.t. "x"

$$2y\frac{dy}{dx} + 2x \cdot 1 = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

Diff. again w.r.t. "x"

$$\frac{d^2y}{dx^2} = -\left(\frac{y \cdot 1 - x \cdot \frac{dy}{dx}}{y^2}\right)$$

$$\frac{d^2y}{dx^2} = -\left(\frac{y - x(-x/y)}{y^2}\right)$$

$$\frac{d^2y}{dx^2} = -\left(\frac{y^2+x^2}{y^3}\right) = -\frac{a^2}{y^3} \quad \text{Ans}$$

Tahir Mahmood
M.Sc. (Math)
Mob No: 0345-6510779

(ii) $x^3 - y^3 = a^3$

Diff. w.r.t. "x"

$$3x^2 \cdot 1 - 3y^2\frac{dy}{dx} = 0$$

$$3y^2\frac{dy}{dx} = 3x^2$$

$$\frac{dy}{dx} = \frac{x^2}{y^2}$$

Diff. w.r.t. "x"

$$\frac{d^2y}{dx^2} = \frac{y^2 \cdot 2x - x^2 \cdot 2y\, dy/dx}{y^4}$$

$$\frac{d^2y}{dx^2} = \frac{2xy^2 - 2x^2y\left(\frac{x^2}{y^2}\right)}{y^4}$$

$$\frac{d^2y}{dx^2} = \frac{2xy^2 - \frac{2x^4}{y}}{y^4} = \frac{2xy^3 - 2x^4}{y^5}$$

$$\frac{d^2y}{dx^2} = \frac{2x(y^3 - x^3)}{y^5} = \frac{-2xa^3}{y^5}$$

(iii) $x = a\cos\theta$, $y = a\sin\theta$

Diff w.r.t. θ, we get

$\frac{dx}{d\theta} = a(-\sin\theta)$ | $\frac{dy}{d\theta} = a\cdot\cos\theta$

$\frac{dx}{d\theta} = -a\sin\theta$ | $\frac{dy}{d\theta} = a\cos\theta$

Using Chain Rule

$$\frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx}$$

$$\frac{dy}{dx} = a\cos\theta\cdot\frac{-1}{a\sin\theta} = -\frac{\cos\theta}{\sin\theta}$$

$$\Rightarrow \frac{dy}{dx} = -\cot\theta.$$

$$\Rightarrow \frac{d^2y}{dx^2} = -(-\operatorname{cosec}^2\theta)\cdot\frac{d\theta}{dx}$$

$$\frac{d^2y}{dx^2} = -(-\operatorname{cosec}^2\theta)\cdot{}^{-1}/{a\sin\theta}$$

$$\frac{d^2y}{dx^2} = \frac{-1}{a}\operatorname{cosec}^3\theta \quad \text{Ans}$$

(vi) $x = at^2$, $y = bt^4$

Diff. w.r.t. "t"

$\frac{dx}{dt} = 2at$ $\qquad \frac{dy}{dt} = 4bt^3$

Using Chain Rule

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}$$

$$\frac{dy}{dx} = 4bt^3\cdot\frac{1}{2at} = \frac{2b}{a}t^2$$

Diff. again w.r.t. "x"

$$\frac{d^2y}{dx^2} = \frac{2b}{a}\cdot 2t\cdot\frac{dt}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{4bt}{a}\cdot\frac{1}{2at} \quad \text{Ans.}$$

$$= \frac{2b}{a^2} \quad \text{Ans.}$$

(v) $x^2+y^2+2gx+2fy+c=0$

Diff w.r.t. "x"

$$2x+2y\frac{dy}{dx}+2g\cdot 1+2f\frac{dy}{dx}+0=0$$

$$(2x+2g)+\left(2y\frac{dy}{dx}+2f\frac{dy}{dx}\right)=0$$

$$2\frac{dy}{dx}(y+f) = -2(x+g)$$

$$\frac{dy}{dx} = \frac{-2(x+g)}{2(y+f)} = -\frac{(x+g)}{(y+f)}$$

Diff. again w.r.t. "x"

$$\frac{d^2y}{dx^2} = -\frac{d}{dx}\left(\frac{x+g}{y+f}\right)$$

$$\frac{d^2y}{dx^2} = -\left[\frac{(y+f)(1+0)-(x+g)\left(\frac{dy}{dx}+0\right)}{(y+f)^2}\right]$$

$$\frac{d^2y}{dx^2} = -\left(\frac{(y+f)-(x+g)\left(\frac{-(x+g)}{(y+f)}\right)}{(y+f)^2}\right)$$

$$\frac{d^2y}{dx^2} = -\left[\frac{(y+f)^2+(x+g)^2}{(y+f)^3}\right] \quad \text{Ans.}$$

Q.4 Find y_4 if:

(i) $y = \sin 3x$

Diff. w.r.t. x 4 times

$y_1 = \cos 3x\cdot 3 = 3\cos 3x$

$y_2 = 3(-\sin 3x)\cdot 3 = -9\sin 3x$

$y_3 = -9(\cos 3x)\cdot 3 = -27\cos 3x$

$y_4 = -27(-\sin 3x)\cdot 3 = 81\sin 3x$

$\boxed{y_4 = 81\sin 3x}$ Ans.

(ii) $y = \cos^3 x$

Using $\cos 3x = 4\cos^3 x - 3\cos x$

$\Rightarrow \cos^3 x = \frac{1}{4}(\cos 3x + 3\cos x)$

$y = \frac{1}{4}[\cos 3x + 3\cos x]$

Diff w.r.t. x, 4 times.

$y_1 = \frac{1}{4}(-\sin 3x \cdot 3 - 3 \cdot \sin x)$

$y_1 = \frac{1}{4}[-3\sin 3x - 3\sin x]$

$y_2 = \frac{1}{4}[-3(\cos 3x \cdot 3) - 3(\cos x)]$

$y_2 = \frac{1}{4}[-9\cos 3x - 3\cos x]$

$y_3 = \frac{1}{4}[-9(-\sin 3x \cdot 3) - 3(-\sin x)]$

$y_3 = \frac{1}{4}[27\sin 3x + 3\sin x]$

$y_4 = \frac{1}{4}[27(\cos 3x \cdot 3) + 3(\cos x)]$

$y_4 = \frac{1}{4}[81\cos 3x + 3\cos x]$

$y_4 = \frac{3}{4}[27\cos 3x + \cos x]$ Ans.

(iii) $y = \ln(x^2 - 9)$ TAHIR

$y = \ln(x+3)(x-3)$ using Ist Law of Log.

$y = \ln(x+3) + \ln(x-3)$

Diff. w.r.t "x" 4 times

$y_1 = \frac{1}{x+3} + \frac{1}{x-3} = (x+3)^{-1} + (x-3)^{-1}$

$y_2 = -1(x+3)^{-2} - 1(x-3)^{-2}$

$y_3 = 2(x+3)^{-3} + 2(x-3)^{-3}$

$y_3 = 2(-3)(x+3)^{-4} + 2(-3)(x-3)^{-4}$

$y_4 = \frac{-6}{(x+3)^4} - \frac{6}{(x-3)^4}$ Ans.

Q.5 If $x = \sin\theta$, $y = \sin(m\theta)$ then Prove that $(1-x^2)y_2 - xy_1 + m^2 y = 0$

Proof:- $x = \sin\theta \Rightarrow \theta = \sin^{-1}x$

(2) $\Rightarrow$ $y = \sin m(\sin^{-1}x)$

$\sin^{-1}y = m\sin^{-1}x$

Diff. w.r.t, x

$\frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = m\frac{1}{\sqrt{1-x^2}}$

Squaring both sides,

$\frac{y_1^2}{(1-y^2)} = \frac{m^2}{(1-x^2)}$ where $y_1 = \frac{dy}{dx}$

$(1-x^2)y_1^2 = m^2(1-y^2)$

Diff. again w.r.t "x"

$(1-x^2) \cdot 2y_1y_2 + y_1^2(-2x) = m^2(0 - 2yy_1)$

$2y_1y_2(1-x^2) - 2xy_1^2 + 2m^2yy_1 = 0$

$2y_1[(1-x^2)y_2 - xy_1 + m^2 y] = 0$

$(1-x^2)y_2 - xy_1 + m^2y = 0$ Ans. $\because 2y_1 \neq 0$

Q.6 If $y = e^x \sin x$ then prove

$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$ TAHIR

Proof: $y = e^x \sin x$ —— ①

Diff. w.r.t. x, we have

$\frac{dy}{dx} = e^x \cdot \cos x \cdot 1 + e^x \cdot 1 \cdot \sin x$

$dy/dx = e^x\cos x + e^x \sin x$ —— ②

Diff. w.r.t. x again

$\frac{d^2y}{dx^2} = [e^x \cdot \cos x + e^x(-\sin x)] + [e^x \cdot \cos x + e^x \cdot 1 \sin x]$

$\frac{d^2y}{dx^2} = e^x\cos x - \cancel{e^x\sin x} + e^x\cos x + \cancel{e^x\sin x}$

$$\frac{d^2y}{dx^2} = 2e^x \cos x$$

Multiply ② by 2 and Subtracting

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 2e^x \cos x - 2\frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 2e^x \cos x - 2(e^x \cos x + e^x \sin x)$$

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 2e^x \cancel{\cos x} - 2e^x \cancel{\cos x} - 2e^x \sin x$$

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = -2e^x \sin x$$

Multiplying ① by 2 and adding

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = -2e^x \sin x + 2y$$

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = -2e^x \cancel{\sin x} + 2e^x \cancel{\sin x}$$

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0 \text{ (Proved).}$$

Q.7 If $y = e^{ax} \cdot \sin bx$ then show that

$$\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2+b^2)y = 0$$

Proof: $y = e^{ax} \cdot \sin bx$ —— ①

Diff. w.r.t x, we have

$$\frac{dy}{dx} = e^{ax} \cdot a \sin bx + e^{ax} \cos bx \cdot b.$$

$$dy/dx = a e^{ax} \sin bx + b e^{ax} \cos bx \text{ —— ②}$$

Diff. again w.r.t. "x"

$$\frac{d^2y}{dx^2} = a[e^{ax} \cdot a \sin bx + e^{ax} \cos bx \cdot b] + b[e^{ax} \cdot a \cos bx + e^{ax}(-\sin bx) b]$$

$$\frac{d^2y}{dx^2} = a^2 e^{ax} \sin bx + ab e^{ax} \cos bx + ab e^{ax} \cos bx - b^2 e^{ax} \sin bx$$

$$\frac{d^2y}{dx^2} = (a^2 - b^2) e^{ax} \sin bx + 2ab e^{ax} \cos bx$$

$$\frac{d^2y}{dx^2} = (a^2-b^2)y + 2ab e^{ax} \cos bx$$

Multiplying ② by $2a$ and Subtracting

$$\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} = (a^2-b^2)y + 2ab e^{ax} \cos bx - 2a[a e^{ax} \sin bx + b e^{ax} \cos bx]$$

$$\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} = (a^2-b^2)y + 2ab e^{ax} \cos bx - 2a[ay + b e^{ax} \cos bx]$$

$$= (a^2-b^2)y - 2a^2y - \cancel{2ab e^{ax} \cos bx} + \cancel{2ab e^{ax} \cos bx}$$

$$\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} = (a^2 - b^2 - 2a^2)y$$

$$\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} = -(a^2+b^2)y$$

$$\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2+b^2)y = 0$$

(Proved).

Q.8 If $y = (\cos^{-1} x)^2$ then prove $(1-x^2)y_2 - xy_1 - 2 = 0$?

Proof: $y = (\cos^{-1} x)^2$

Diff w.r.t. x

$$y_1 = 2\cos^{-1}x \cdot \frac{-1}{\sqrt{1-x^2}}$$

$$y_1 = \frac{-2\cos^{-1}x}{\sqrt{1-x^2}}$$

Squaring both sides.

$$y_1^2 = \frac{4(\cos^{-1}x)^2}{(1-x^2)}$$

$$(1-x^2)y_1^2 = 4y \qquad \because y = (\cos^{-1}x)^2$$

Diff. again w.r.t. x

$$(1-x^2)2y_1y_2-2xy_1^2=4y_1$$

$$2y_1\left[(1-x^2)y_2-xy_1\right]=4y_1$$

$$(1-x^2)y_2-xy_1=2$$

$$(1-x^2)y_2-xy_1-2=0 \text{ (Proved)}$$

Q.9 If $y=a\cos(\ln x)+b\sin(\ln x)$

Prove that $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0$

Proof:: $y=a\cos(\ln x)+b\sin(\ln x)$

Diff. w.r.t. x

$$\frac{dy}{dx}=a\frac{(-\sin(\ln x))}{x}+b\frac{\cos(\ln x)}{x}$$

$$\frac{dy}{dx}=\frac{-a\sin(\ln x)+b\cos(\ln x)}{x}$$

$$x\frac{dy}{dx}=-a\sin(\ln x)+b\cos(\ln x)$$

Diff. again w.r.t "x"

$$x\frac{d^2y}{dx^2}+1.\frac{dy}{dx}=-\frac{a\cos(\ln x)}{x}-\frac{b\sin(\ln x)}{x}$$

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}=\frac{-a\cos(\ln x)-b\sin(\ln x)}{x}$$

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}=-(a\cos(\ln x)+b\sin(\ln x))$$

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}=-y$$

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0 \text{ (Proved).}$$

Maclaurin Series:-

If $f(x)$ is a real valued and Continous function then

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\cdots$$

is called the Maclaurin's Series. This Series was introduced by Maclaurins in 1742.

Taylor's Series:-

If f is a real valued and Continuous function then

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+\cdots$$

is called Taylor's Series of $f(x+h)$. This Series was introduced by a British Mathematician Taylor.

Exercise 2.8

Q.1 Apply Maclaurins Series to prove that:

(i) $\ln(x+1)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots\cdots$

Let $f(x)=\ln(x+1)$

$f(x)=\ln(x+1) \Rightarrow f(0)=\ln(1+0)=0$

$f'(x)=\frac{1}{x+1}=(x+1)^{-1} \Rightarrow f'(0)=(0+1)^{-1}=1$

$f''(x)=-1(x+1)^{-2} \Rightarrow f''(0)=-1$

$f'''(x)=2(x+1)^{-3} \Rightarrow f'''(0)=2$

$\vdots \qquad \vdots$