Exercise 2.6

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(26)

Q.1 Find $f'(x)$ if:

(i) $f(x) = e^{\sqrt{x}-1}$

Diff. w.r.t. x

$$f'(x) = \frac{d}{dx}\left(e^{\sqrt{x}-1}\right)$$

$$f'(x) = e^{\sqrt{x}-1} \cdot \frac{d}{dx}\left(\sqrt{x}-1\right)$$

$$f'(x) = e^{\sqrt{x}-1}\left(\frac{1}{2\sqrt{x}} - 0\right)$$

$$f'(x) = \frac{e^{\sqrt{x}-1}}{2\sqrt{x}} \quad \text{Ans.}$$

(ii) $f(x) = x^3 e^{1/x} \qquad (x \neq 0)$

Diff. w.r.t. x

$$f'(x) = x^3 \frac{d}{dx}\left(e^{1/x}\right) + e^{1/x} \cdot \frac{d}{dx}(x^3)$$

$$f'(x) = x^3 \cdot e^{1/x} \frac{d}{dx}\left(1/x\right) + e^{1/x} \cdot 3x^2 \cdot 1$$

$$f'(x) = x^3 e^{1/x}\left[\frac{x \cdot 0 - 1 \cdot 1}{x^2}\right] + 3x^2 e^{1/x}$$

$$f'(x) = x^3 e^{1/x}\left(\frac{-1}{x^2}\right) + 3x^2 e^{1/x}$$

$$f'(x) = -x\, e^{1/x} + 3x^2 e^{1/x}$$

$$f'(x) = x e^{1/x}\left[3x - 1\right] \quad \text{Ans.}$$

(iii) $f(x) = e^x\left[1 + \ln x\right]$

Diff. w.r.t. x

$$f'(x) = e^x \frac{d}{dx}(1 + \ln x) + (1 + \ln x)\frac{d}{dx}(e^x)$$

$$f'(x) = e^x\left(0 + \frac{1}{x}\right) + (1 + \ln x)\, e^x \cdot 1$$

$$f'(x) = \frac{e^x}{x} + (1 + \ln x)\, e^x$$

$$f'(x) = e^x\left[\frac{1}{x} + 1 + \ln x\right]$$

$$f'(x) = e^x\left[\frac{1 + x + x\ln x}{x}\right] \quad \text{Ans.}$$

(iv) $f(x) = \dfrac{e^x}{e^{-x}+1}$

Diff. w.r.t. "x"

$$f'(x) = \frac{(e^{-x}+1)\frac{d}{dx}(e^x) - e^x \frac{d}{dx}(e^{-x}+1)}{(e^{-x}+1)^2}$$

$$f'(x) = \frac{(e^{-x}+1)\, e^x \cdot 1 - e^x\left[e^{-x}(-1) + 0\right]}{(e^{-x}+1)^2}$$

$$f'(x) = \frac{e^{-x} \cdot e^x + e^x + e^{-x} \cdot e^x}{(e^{-x}+1)^2}$$

$$f'(x) = \frac{e^0 + e^x + e^0}{(e^{-x}+1)^2} = \frac{1 + e^x + 1}{(e^{-x}+1)^2}$$

$$f'(x) = \frac{2 + e^x}{(e^{-x}+1)^2} \quad \text{Ans.}$$

(v) $f(x) = \ln\left[e^x + e^{-x}\right]$

Diff. w.r.t. "x"

$$f'(x) = \frac{1}{e^x + e^{-x}} \cdot \frac{d}{dx}\left[e^x + e^{-x}\right]$$

$$f'(x) = \frac{1}{(e^x + e^{-x})} \cdot e^x + e^{-x}(-1)$$

$$f'(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} \quad \text{Ans.}$$

(vi) $f(x) = \dfrac{e^{ax} - e^{-ax}}{e^{ax} + e^{-ax}}$

Diff. w.r.t. "x"

$$f'(x) = \frac{d}{dx}\left[\frac{e^{ax} - e^{-ax}}{e^{ax} + e^{-ax}}\right]$$

$$f'(x) = \frac{(e^{ax}+e^{-ax})\frac{d}{dx}(e^{ax}-e^{-ax}) - (e^{ax}-e^{-ax})\frac{d}{dx}(e^{ax}+e^{-a}}{(e^{ax}+e^{-ax})^2}$$

$$f'(x) = \frac{(e^{ax}+e^{-ax})(e^{ax} \cdot a + e^{-ax} \cdot a) - (e^{ax}-e^{-ax})(e^{ax} \cdot a - a e^{ax}}{(e^{ax}+e^{-ax})^2}$$

$$f'(x) = \frac{a\left[e^{ax} + e^{-ax}\right]^2 - a\left[e^{ax} - e^{-ax}\right]^2}{(e^{ax}+e^{-ax})^2}$$

$$f'(x)=\frac{a(\cancel{e^{2ax}}+\cancel{e^{-2ax}}+2-\cancel{e^{2ax}}-\cancel{e^{-2ax}}+2)}{(e^{ax}+e^{-xa})^2}$$

$$f'(x)=\frac{a(4)}{(e^{ax}+e^{-ax})^2}=\frac{4a}{(e^{ax}+e^{-ax})^2}$$ Ans.

(vii) $f(x)=\sqrt{\ln(e^{2x}+e^{-2x})}$.

Diff. w.r.t. "x"

$$f'(x)=\frac{d}{dx}\left[\sqrt{\ln(e^{2x}+e^{-2x})}\right]$$

$$f'(x)=\frac{1}{2\sqrt{\ln(e^{2x}+e^{-2x})}}\cdot\frac{d}{dx}\left[\ln(e^{2x}+e^{-2x})\right]$$

$$f'(x)=\frac{1}{2\sqrt{\ln(e^{2x}+e^{-2x})}}\cdot\frac{1}{(e^{2x}+e^{-2x})}(2e^{2x}-2e^{-2x})$$

$$f'(x)=\frac{\cancel{2}(e^{2x}-e^{-2x})}{\cancel{2}(e^{2x}+e^{-2x})\sqrt{\ln(e^{2x}+e^{-2x})}}$$

$$f'(x)=\frac{(e^{2x}-e^{-2x})}{(e^{2x}+e^{-2x})\sqrt{\ln(e^{2x}+e^{-2x})}}$$ Ans.

(viii) $f(x)=\ln\left(\sqrt{e^{2x}+e^{-2x}}\right)$

Diff. w.r.t. "x"

$$f'(x)=\frac{d}{dx}\left[\ln\sqrt{e^{2x}+e^{-2x}}\right]$$

$$f'(x)=\frac{d}{dx}\left[\ln(e^{2x}+e^{-2x})^{1/2}\right]$$

$$f'(x)=\frac{1}{2}\frac{d}{dx}\left(\ln(e^{2x}+e^{-2x})\right)$$

$$f'(x)=\frac{1}{2}\cdot\frac{1}{e^{2x}+e^{-2x}}\cdot 2e^{2x}-2e^{-2x}$$

$$f'(x)=\frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}$$ Ans.

Q.2 Find $\frac{dy}{dx}=?$ if:

(i) $y=x^2\ln\sqrt{x}$

Diff. w.r.t. "x"

$$\frac{dy}{dx}=x^2\frac{d}{dx}(\ln\sqrt{x})+\ln\sqrt{x}\cdot\frac{d}{dx}(x^2)$$

$$\frac{dy}{dx}=\frac{x^2}{\sqrt{x}}\cdot\frac{1}{2\sqrt{x}}+\ln\sqrt{x}\cdot(2x)$$

$$\frac{dy}{dx}=\frac{x^2}{2x}+2x\ln\sqrt{x}$$

$$\frac{dy}{dx}=\frac{x}{2}+2x\ln\sqrt{x}=x\left(\frac{1}{2}+2\ln\sqrt{x}\right)$$ Ans.

(ii) $y=x\cdot\sqrt{\ln x}$

Diff. w.r.t. "x"

$$\frac{dy}{dx}=x\frac{d}{dx}(\sqrt{\ln x})+\sqrt{\ln x}\,\frac{d}{dx}(x)$$

$$\frac{dy}{dx}=\frac{\cancel{x}}{2\sqrt{\ln x}}\cdot\frac{1}{\cancel{x}}+\sqrt{\ln x}\cdot 1$$

$$\frac{dy}{dx}=\frac{1}{2\sqrt{\ln x}}+\sqrt{\ln x}$$

$$\frac{dy}{dx}=\frac{1+2\ln x}{2\sqrt{\ln x}}$$ Ans.

(iii) $y=\frac{x}{\ln x}$

Diff. w.r.t. "x"

$$\frac{dy}{dx}=\frac{(\ln x)\frac{d(x)}{dx}-x\frac{d}{dx}(\ln x)}{(\ln x)^2}$$

$$\frac{dy}{dx}=\frac{\ln x\cdot 1-x\cdot 1/x}{(\ln x)^2}$$

$$\frac{dy}{dx}=\frac{\ln x-1}{(\ln x)^2}$$ Ans.

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(iv) $y = x^2 \ln\left(\frac{1}{x}\right)$

$y = x^2 \ln[x^{-1}]$ $\quad (\because \log m^n = n \log m)$

$y = x^2(-\ln x)$

$y = -x^2 \ln x$

Diff. w.r.t "x"

$$\frac{dy}{dx} = -\left(\frac{d}{dx}(x^2)\ln x + x^2 \frac{d}{dx}(\ln x)\right)$$

$$\frac{dy}{dx} = -\left(2x\ln x + x^2 \cdot \frac{1}{x}\right)$$

$$\frac{dy}{dx} = -(2x\ln x + x)$$

$$\frac{dy}{dx} = -x[1 + 2\ln x] \text{ Ans.}$$

(v) $y = \ln\sqrt{\frac{x^2-1}{x^2+1}}$

Diff. w.r.t. "x"

$$\frac{dy}{dx} = \frac{d}{dx}\left(\ln\sqrt{\frac{x^2-1}{x^2+1}}\right)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{\frac{x^2-1}{x^2+1}}} \cdot \frac{d}{dx}\left(\frac{x^2-1}{x^2+1}\right)^{1/2}$$

$$\frac{dy}{dx} = \frac{\sqrt{x^2+1}}{\sqrt{x^2-1}} \cdot \frac{1}{2}\left(\frac{x^2-1}{x^2+1}\right)^{-1/2} \frac{d}{dx}\left(\frac{x^2-1}{x^2+1}\right)$$

$$\frac{dy}{dx} = \frac{1}{2}\frac{\sqrt{x^2+1}}{\sqrt{x^2-1}}\left(\frac{x^2+1}{x^2-1}\right)^{1/2} \cdot \frac{(x^2+1)(2x-0)-(x^2-1)(2x+0)}{(x^2+1)^2}$$

$$\frac{dy}{dx} = \frac{1}{2}\frac{\cancel{(x^2+1)}}{(x^2-1)} \cdot \frac{\cancel{2x^3}+2x-\cancel{2x^3}+2x}{(x^2+1)^{\cancel{2}}}$$

$$\frac{dy}{dx} = \frac{4x}{2(x^2-1)(x^2+1)}$$

$$\frac{dy}{dx} = \frac{2x}{(x^4-1)} \text{ Ans.}$$

(vi) $y = \ln(x + \sqrt{x^2+1})$

Diff. w.r.t. "x", we have

$$\frac{dy}{dx} = \frac{d}{dx}\left[\ln(x+\sqrt{x^2+1})\right]$$

$$\frac{dy}{dx} = \frac{1}{(x+\sqrt{x^2+1})} \cdot \frac{d}{dx}\left[x+\sqrt{x^2+1}\right]$$

$$\frac{dy}{dx} = \frac{1}{(x+\sqrt{x^2+1})} \cdot 1 + \frac{1}{\cancel{2}\sqrt{x^2+1}} \cdot \cancel{2}x$$

$$\frac{dy}{dx} = \frac{\cancel{(\sqrt{x^2+1}+x)}}{\sqrt{x^2+1}\,\cancel{(x+\sqrt{x^2+1})}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2+1}} \text{ Ans.}$$

(vii) $y = \ln[9-x^2]$

Diff. w.r.t. "x"

$$\frac{dy}{dx} = \frac{d}{dx}\left[\ln(9-x^2)\right]$$

$$\frac{dy}{dx} = \frac{1}{9-x^2} \cdot (0-2x)$$

$$\frac{dy}{dx} = \frac{-2x}{9-x^2} \text{ Ans.}$$

(viii) $y = e^{-2x} \cdot \sin 2x$

Diff. w.r.t. "x"

$$\frac{dy}{dx} = \frac{d}{dx}(e^{-2x}) \cdot \sin 2x + e^{-2x} \cdot \frac{d}{dx}(\sin 2x)$$

$$\frac{dy}{dx} = -2e^{-2x}\sin 2x + e^{-2x} \cdot \cos 2x \cdot 2$$

$$\frac{dy}{dx} = 2e^{-2x}[\cos 2x - \sin 2x] \text{ Ans.}$$

(ix) $y = e^{-x}(x^3 + 2x^2 + 1)$

Diff. w.r.t. "x"

$$\frac{dy}{dx} = \frac{d}{dx}(e^{-x}) \cdot (x^3+2x^2+1) + e^{-x} \cdot \frac{d}{dx}(x^3+2x^2+1)$$

$$\frac{dy}{dx} = -e^{-x}(x^3+2x^2+1) + e^{-x}(3x^2+4x+0)$$

$$\frac{dy}{dx} = e^{-x}[3x^2+4x-x^3-2x^2-1]$$

$$\frac{dy}{dx} = e^{-x}(x^2+4x-x^3-1) \text{ Ans.}$$

(x) $y = xe^{\sin x}$

Diff. w.r.t. "x"

$$\frac{dy}{dx} = \frac{d}{dx}(x)\cdot e^{\sin x} + x\,\frac{d}{dx}(e^{\sin x})$$

$$\frac{dy}{dx} = 1\cdot e^{\sin x} + x\cdot e^{\sin x}\cos x$$

$$dy/dx = e^{\sin x}\left(1 + x\cos x\right) \text{ Ans.}$$

(xi) $y = 5e^{3x-4}$

Diff. w.r.t. "x"

$$\frac{dy}{dx} = 5\,\frac{d}{dx}\left[e^{3x-4}\right]$$

$$dy/dx = 5e^{3x-4}(3.1-0)$$

$$dy/dx = 15e^{3x-4} \text{ Ans.}$$

Imp. (xii) $y = (x+1)^x$

Taking ln on both Sides.

$$\ln y = \ln(x+1)^x$$

$$\ln y = x\ln(x+1)$$

Diff. w.r.t. "x"

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}(x)\cdot\ln(x+1) + x\cdot\frac{d}{dx}(\ln(x+1))$$

$$\frac{1}{y}\frac{dy}{dx} = 1\cdot\ln(x+1) + x\cdot\frac{1}{x+1}\cdot 1$$

$$\frac{dy}{dx} = y\left[\ln(x+1) + \frac{x}{x+1}\right]$$

$$\frac{dy}{dx} = (x+1)^x\left[\ln(x+1) + \frac{x}{x+1}\right] \quad (\because y = (x+1)^x)$$

(xiii) $y = (\ln x)^{\ln x}$

Taking ln on both Sides.

$$\ln y = \ln[\ln x]^{\ln x}$$

$$\ln y = (\ln x)\cdot(\ln(\ln x))$$

Diff. w.r.t. "x"

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}(\ln x)\cdot\ln(\ln x) + (\ln x)\frac{d}{dx}(\ln(\ln x))$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{\ln(\ln x)}{x} + \cancel{\ln x}\cdot\frac{1}{\cancel{\ln x}}\cdot\frac{1}{x}$$

$$\frac{1}{y}\frac{dy}{dx} = \left(\frac{\ln(\ln x)}{x} + \frac{1}{x}\right)$$

$$\frac{dy}{dx} = \frac{y}{x}\left[\ln(\ln x) + 1\right]$$

$$\frac{dy}{dx} = \frac{(\ln x)^{\ln x}}{x}\left[1 + \ln(\ln x)\right] \text{ Ans.}$$

(xiv) $y = \dfrac{\sqrt{x^2-1}\,(x+1)}{(x^3+1)^{3/2}}$

Taking "ln" on both Sides

$$\ln y = \ln\frac{\sqrt{x^2-1}\,(x+1)}{(x^3+1)^{3/2}}$$

Using Laws of Logarithm

$$\ln y = \ln\sqrt{x^2-1} + \ln(x+1) - \ln(x^3+1)^{3/2}$$

$$\ln y = \frac{1}{2}\ln(x^2-1) + \ln(x+1) - \frac{3}{2}\ln(x^3+1)$$

Diff. w.r.t. x

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{\cancel{2}}\frac{\cancel{2}x}{(x^2-1)} + \frac{1.1}{x+1} - \frac{3}{2}\,\frac{(3x^2+0)}{x^3+1}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{x}{x^2-1} + \frac{1}{x+1} - \frac{9x^2}{2(x^3+1)}$$

$$\frac{dy}{dx} = y\left[\frac{x}{x^2-1} + \frac{1}{x+1} - \frac{9x^2}{2(x^3+1)}\right]$$

Putting the Value of y

$$\frac{dy}{dx} = \frac{\sqrt{x^2-1}\,(x+1)}{(x^3+1)^{3/2}}\left[\frac{x}{x^2-1} + \frac{1}{x+1} - \frac{9x^2}{2(x^3+1)}\right]$$

Ans.

Q.3 Find $\dfrac{dy}{dx}$ if:

(i) $y = \cosh 2x$

Diff. w.r.t. "x"

$$\frac{dy}{dx} = \frac{d}{dx}\left[\cosh 2x\right]$$

$$\frac{dy}{dx} = \sinh 2x\cdot 2 = 2\sinh 2x.$$

Ans.

(ii) $y = \sinh 3x$

Diff. w.r.t "x"

$$\frac{dy}{dx} = \frac{d}{dx}[\sinh 3x]$$

$$\frac{dy}{dx} = \cosh 3x \cdot 3$$

$$\frac{dy}{dx} = 3\cosh 3x \quad \text{Ans.}$$

(iii) $y = \tanh^{-1}(\sin x)$

Diff. w.r.t. "x"

$$\frac{dy}{dx} = \frac{d}{dx}[\tanh^{-1}(\sin x)]$$

$$\frac{dy}{dx} = \frac{1}{1-(\sin x)^2} \cdot \cos x$$

$$\frac{dy}{dx} = \frac{1}{1-\sin^2 x} \cdot \cos x$$

$$\frac{dy}{dx} = \frac{\cos x}{\cos^2 x} = \frac{1}{\cos x}$$

$$\frac{dy}{dx} = \sec x \quad \text{Ans.}$$

(iv) $y = \sinh^{-1}(x^3)$

Diff. w.r.t. "x"

$$\frac{dy}{dx} = \frac{d}{dx}[\sinh^{-1}(x^3)]$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+(x^3)^2}} \cdot 3x^2$$

$$\frac{dy}{dx} = \frac{3x^2}{\sqrt{1+x^6}} \quad \text{Ans.}$$

(v) $y = (\ln \tanh^{-1} x)$

Diff. w.r.t. "x"

$$\frac{dy}{dx} = \frac{d}{dx}(\ln \tanh^{-1} x)$$

$$\frac{dy}{dx} = \frac{1}{\tanh^{-1} x} \cdot \frac{1}{1-x^2} = \frac{1}{(1-x^2)\tanh^{-1} x}$$

Ans.

(vi) $y = \sinh^{-1}\left(\frac{x}{2}\right)$

Diff. w.r.t. "x"

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sinh^{-1}\left(\frac{x}{2}\right)\right)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+\left(\frac{x}{2}\right)^2}} \cdot \frac{d}{dx}\left(\frac{x}{2}\right)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{\frac{4+x^2}{4}}} \cdot \frac{1}{2}$$

$$\frac{dy}{dx} = \frac{\sqrt{4}}{\sqrt{4+x^2}} \cdot \frac{1}{2} = \frac{\cancel{2}}{\sqrt{4+x^2}} \cdot \frac{1}{\cancel{2}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{4+x^2}} \quad \text{Ans.}$$

Higher Derivatives:-

If $y = f(x)$ is a real valued and continuous function then $\frac{dy}{dx} = f'(x)$ is its first derivative and $\frac{d^2y}{dx^2} = f''(x)$ is called 2nd derivative of $y = f(x)$ and $\frac{d^3y}{dx^3} = f'''(x)$ is called 3rd Derivative of $y = f(x)$ and so on.

Higher derivatives are also named as Consecutive or Successive derivatives.

In fact $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left[\frac{d^2y}{dx^2}\right]$$