Exercise 2.5

Q1: Differentiate the followings by first Principle:

(i) $\sin 2x$

Let $y = \sin 2x$ ——(1)

$y + \delta y = \sin 2(x+\delta x)$ ——(2)

Subtracting Eq (1) from Eq (2)

$\delta y = \sin(2x+2\delta x) - \sin 2x$

$\delta y = 2\cos\left(\frac{2x+2\delta x+2x}{2}\right).\sin\left(\frac{2x+2\delta x-2x}{2}\right)$

$\delta y = 2\cos\left(\frac{4x+2\delta x}{2}\right).\sin\left(\frac{2\delta x}{2}\right)$

$\delta y = 2\cos(2x+\delta x).\sin\delta x$

$\because \sin P - \sin Q = 2\cos\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)$

Dividing both sides by δx where $\lim\limits_{\delta x\to 0}$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{2\cos(2x+\delta x)\cdot\sin\delta x}{\delta x}$$

$$\frac{dy}{dx} = 2.\lim_{\delta x\to 0}\cos(2x+\delta x).\lim_{\delta x\to 0}\frac{\sin\delta x}{\delta x}$$

$$\frac{dy}{dx} = 2\cos(2x+0).\ 1$$

$$\boxed{\frac{dy}{dx} = 2\cos 2x} \quad \text{Ans.}$$

$\left(\because \lim\limits_{x\to 0}\frac{\sin x}{x} = 1\right)$

(ii) $\tan 3x$

Let $y = \tan 3x$ ——(1)

$y + \delta y = \tan 3(x+\delta x)$ ——(2)

Subtracting Eq (1) from Eq (2)

$\delta y = \tan(3x+3\delta x) - \tan 3x$

$\delta y = \frac{\sin(3x+3\delta x)}{\cos(3x+\delta x\cdot 3)} - \frac{\sin 3x}{\cos 3x}$

$\delta y = \frac{\sin(3x+3\delta x)\cos 3x - \cos(3x+3\delta x).\sin 3x}{\cos(3x+3\delta x).\cos 3x}$

$\delta y = \frac{\sin(\cancel{3x}+3\delta x-\cancel{3x})}{\cos 3x\cdot\cos(3x+3\delta x)}$

$(\because \sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta)$

$\delta y = \frac{\sin 3\delta x}{\cos 3x\cdot\cos(3x+3\delta x)}$

Dividing both sides by δx where $\lim\limits_{\delta x\to 0}$

$$\operatorname*{Lt}_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{\sin 3\delta x}{\delta x\cdot\cos 3x\cdot\cos(3x+3\delta x)}$$

$$\frac{dy}{dx} = 3.\operatorname*{Lt}_{\delta x\to 0}\frac{\sin 3\delta x}{3\delta x}.\operatorname*{Lt}_{\delta x\to 0}\frac{1}{\cos 3x.\cos(3x+3\delta x)}$$

$$\frac{dy}{dx} = 3.1.\frac{1}{\cos 3x\cdot\cos(3x+0)}$$

$$\frac{dy}{dx} = \frac{3}{\cos^2 3x}$$

$$\boxed{\frac{dy}{dx} = 3\sec^2 3x} \quad \text{Ans.}$$

$\left(\because \lim\limits_{x\to 0}\frac{\sin x}{x} = 1\right)$

(iii) $\sin 2x + \cos 2x$

Let $y = \sin 2x + \cos 2x$ ——(1)

$y + \delta y = \sin 2(x+\delta x) + \cos 2(x+\delta x)$ ——(2)

Subtracting Eq (1) from Eq (2)

$\delta y = \sin(2x+2\delta x) + \cos(2x+2\delta x) - \sin 2x - \cos 2x$

$\delta y = [\sin(2x+2\delta x) - \sin 2x] + [\cos(2x+2\delta x) - \cos 2x]$

$\delta y = 2\cos\left(\frac{2x+2\delta x+2x}{2}\right)\sin\left(\frac{2x+2\delta x-2x}{2}\right)$

$+ -2\sin\left(\frac{2x+2\delta x+2x}{2}\right)\sin\left(\frac{2x+\delta x\cdot 2-2x}{2}\right)$

$\delta y = 2\cos\left(\frac{4x+2x}{2}\right)\sin\frac{2\delta x}{2} - 2\sin\left(\frac{4x+2\delta x}{2}\right)\sin\left(\frac{2\delta x}{2}\right)$

$\delta y = 2\cos(2x+\delta x)\sin\delta x - 2\sin(2x+\delta x).\sin\delta x$

$\delta y = 2\sin\delta x[\cos(2x+\delta x) - \sin(2x+\delta x)]$

Dividing by δx where $\operatorname*{Lt}\ \delta x\to 0$

$$\operatorname*{Lt}_{\delta x\to 0}\frac{\delta y}{\delta x} = \operatorname*{Lt}_{\delta x\to 0}\frac{2\sin\delta x(\cos(2x+\delta x) - \sin(2x+\delta x))}{\delta x}$$

$$\frac{dy}{dx} = 2.\operatorname*{Lt}_{\delta x\to 0}\frac{\sin\delta x}{\delta x}.\operatorname*{Lt}_{\delta x\to 0}\cos(2x+\delta x) - \sin(2x+\delta x)$$

$$\frac{dy}{dx} = 2\cdot 1\ \left[Cos(2x+0) - Sin(2x+0)\right]$$

$$\boxed{\frac{dy}{dx} = 2\left[Cos 2x - Sin 2x\right]} \quad \text{Ans.}$$

(iv) $\tan^2 x$

Let $y = \tan^2 x$ ——(1)

$y+\delta y = \tan^2(x+\delta x)$ ——(2)

Subtracting Eq (1) from Eq (2)

$$\delta y = \tan^2(x+\delta x) - \tan^2(x)$$

$$\delta y = \left[\tan(x+\delta x)+\tan x\right]\left[\tan(x+\delta x) - \tan x\right]$$

$$\delta y = (\tan(x+\delta x)+\tan x).\left(\frac{Sin(x+\delta x)}{Cos(x+\delta x)} - \frac{Sin x}{Cos x}\right)$$

$$\delta y = \left[\tan(x+\delta x)+\tan x\right]\left(\frac{Sin(x+\delta x)Cos x - Cos(x+\delta x)Sin x}{Cos(x+\delta x)Cos x}\right)$$

$$\delta y = \left[\tan(x+\delta x)+\tan x\right].\ \frac{Sin(x+\delta x - x)}{Cos(x+\delta x)Cos x}$$

$$\delta y = \left[\tan(x+\delta x)+\tan x\right].\ \frac{Sin\,\delta x}{Cos(x+\delta x)\,Cos x}$$

Dividing by δx $\underset{\delta x \to 0}{Lt}$

$$\underset{\delta x\to 0}{Lt}\frac{\delta y}{\delta x} = \underset{\delta x\to 0}{Lt}\left[\tan(x+\delta x)+\tan x\right].\ \frac{Sin\,\delta x}{\delta x\cdot Cos x\cdot Cos(x+\delta x)}$$

$$\frac{dy}{dx} = \left[\tan(x+0)+\tan x\right].\underset{\delta x\to 0}{Lt}\ \frac{Sin\,\delta x}{\delta x}\ .\ \frac{1}{Cos x\ Cos(x+0)}$$

$$\frac{dy}{dx} = 2\tan x\cdot 1\cdot\ \frac{1}{Cos^2 x}$$

$$\frac{dy}{dx} = 2\tan x\cdot Sec^2 x$$

$$\boxed{\frac{dy}{dx} = 2\tan x\cdot Sec^2 x} \quad \text{Ans.}$$

(v) $Cos\, x^2$

Let $y = Cos x^2$ ——(1)

$y+\delta y = Cos(x+\delta x)^2$ ——(2)

Subtracting Eq(1) from Eq (2)

(19)

$$\delta y = Cos(x+\delta x)^2 - Cos x^2$$

$$\delta y = -2\,Sin\left(\frac{(x+\delta x)^2+x^2}{2}\right).\,Sin\left(\frac{(x+\delta x)^2 - x^2}{2}\right)$$

$$\delta y = -2\,Sin\left(\frac{(x+\delta x)^2+x^2}{2}\right).\,Sin\left(\frac{\cancel{x^2}+\delta x^2+2x\delta x-\cancel{x^2}}{2}\right)$$

$$\delta y = -2\,Sin\left[\frac{(x+\delta x)^2+x^2}{2}\right].\,Sin\left[\delta x\left(\frac{\delta x+2x}{2}\right)\right]$$

Dividing by δx where $\underset{\delta x\to 0}{Lim}$

$$\underset{\delta x\to 0}{Lt}\frac{\delta y}{\delta x} = \underset{\delta x\to 0}{Lt}\ \frac{-2\,Sin\left(\frac{(x+\delta x)^2+x^2}{2}\right)Sin.\,\delta x\left(\frac{\delta x+2x}{2}\right)}{\delta x}$$

$$\frac{dy}{dx} = -2\underset{\delta x\to 0}{Lim}\ Sin\left[\frac{(x+\delta x)^2+x^2}{2}\right]\underset{\delta x\to 0}{Lt}\ \frac{Sin\,\delta x\left(\frac{2x+\delta x}{2}\right)}{\delta x\left(\frac{2x+\delta x}{2}\right)\left(\frac{2}{2x+\delta x}\right)}$$

$$\frac{dy}{dx} = -2\cdot Sin\left(\frac{(x+0)^2+x^2}{2}\right).\ \ 1\cdot\frac{1}{\frac{2}{(2x+0)}}$$

$$\frac{dy}{dx} = -2\,Sin\left(\frac{x^2+x^2}{2}\right).\ x$$

$$\boxed{\frac{dy}{dx} = -2x\,Sin\,x^2} \quad \text{Ans.}$$

(vi) $\sqrt{\tan x}$

Let $y = \sqrt{\tan x}$ ——(1)

$y+\delta y = \sqrt{\tan(x+\delta x)}$ ——(2)

Subtracting Eq (1) from Eq (2)

$$\delta y = \sqrt{\tan(x+\delta x)} - \sqrt{\tan x}$$

By Rationalizing, we have

$$\delta y = \sqrt{\tan(x+\delta x)} - \sqrt{\tan x}\times\frac{\sqrt{\tan(x+\delta x)}+\sqrt{\tan x}}{\sqrt{\tan(x+\delta x)}+\sqrt{\tan x}}$$

$$\delta y = \frac{\left(\sqrt{\tan(x+\delta x)}\right)^2 - \left(\sqrt{\tan x}\right)^2}{\sqrt{\tan(x+\delta x)}+\sqrt{\tan x}}$$

$$\delta y = \frac{\tan(x+\delta x) - \tan x}{\sqrt{\tan(x+\delta x)}+\sqrt{\tan x}}$$

$$\delta y = \frac{\frac{\sin(x+\delta x)}{\cos(x+\delta x)} - \frac{\sin x}{\cos x}}{\sqrt{\tan(x+\delta x)} + \sqrt{\tan x}}$$

$$\delta y = \frac{\sin(x+\delta x)\cos x - \cos(x+\delta x)\sin x}{\sqrt{\tan(x+\delta x)} + \sqrt{\tan x} \quad \cos x \cdot \cos(x+\delta x)}$$

$$\delta y = \frac{\sin(x+\delta x - x)}{\sqrt{\tan(x+\delta x)} + \sqrt{\tan x} \quad \cos x \cdot \cos(x+\delta x)}$$

$$\delta y = \frac{\sin \delta x}{\sqrt{\tan(x+\delta x)} + \sqrt{\tan x} \quad \cos x \cdot \cos(x+\delta x)}$$

Dividing by δx where $\lim_{\delta x \to 0}$

$$\underset{\delta x \to 0}{Lt} \frac{\delta y}{\delta x} = \underset{\delta x \to 0}{Lt} \frac{\sin \delta x}{\delta x(\sqrt{\tan(x+\delta x)} + \sqrt{\tan x})\cos x \cdot \cos(x+\delta x)}$$

$$\frac{dy}{dx} = \underset{\delta x \to 0}{Lt} \frac{\sin \delta x}{\delta x} \cdot \frac{1}{\sqrt{\tan(x+0)} + \sqrt{\tan x} \quad \cos x \cdot \cos(x+0)}$$

$$\frac{dy}{dx} = 1 \cdot \frac{1}{2\sqrt{\tan x} \cdot \cos^2 x}$$

$$\boxed{\frac{dy}{dx} = \frac{\sec^2 x}{2\sqrt{\tan x}}} \quad \text{Ans.}$$

(vii) $\cos\sqrt{x}$

Let $y = \cos\sqrt{x}$ —— (1)

$y + \delta y = \cos\sqrt{x+\delta x}$ —— (2)

Subtracting Eq (1) from Eq (2)

$$\delta y = \cos\sqrt{x+\delta x} - \cos\sqrt{x}$$

$$\delta y = -2\sin\left(\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right)\sin\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)$$

Dividing by δx where $\lim_{\delta x \to 0}$

$$\underset{\delta x \to 0}{Lt} \frac{\delta y}{\delta x} = \underset{\delta x \to 0}{Lt} \frac{-2\sin\left(\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right)\sin\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}{\delta x}$$

$$\because (\sqrt{x+\delta x})^2 - (\sqrt{x})^2 = \delta x$$

$$\delta x = (\sqrt{x+\delta x}+\sqrt{x})(\sqrt{x+\delta x}-\sqrt{x})$$

$$\frac{dy}{dx} = \underset{\delta x \to 0}{Lt} \frac{-2\sin\left(\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right)\sin\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}{2\cdot(\sqrt{x+\delta x}+\sqrt{x})\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}$$

$$\frac{dy}{dx} = \frac{-\cancel{2}\sin\left(\frac{\sqrt{x+0}+\sqrt{x}}{2}\right)}{\cancel{2}(\sqrt{x+0}+\sqrt{x})} \cdot 1$$

$$\frac{dy}{dx} = \frac{-\sin\left(\frac{2\sqrt{x}}{2}\right)}{2\sqrt{x}}$$

$$\boxed{\frac{dy}{dx} = \frac{-1}{2\sqrt{x}}\sin\sqrt{x}} \quad \text{Ans.}$$

Q.2 Differentiate if:

(i) Let $y = x^2 \sec 4x$

Diff. w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}(x^2)\cdot\sec 4x + x^2\frac{d}{dx}(\sec 4x)$$

$$\frac{dy}{dx} = 2x\sec 4x + x^2\cdot\sec 4x\cdot\tan 4x\cdot 4$$

$$\frac{dy}{dx} = 2x\sec 4x[1 + 2x\tan 4x] \quad \text{Ans.}$$

(ii) Let $y = \tan^3\theta\cdot\sec^2\theta$

Diff. w.r.t. θ

$$\frac{dy}{d\theta} = \tan^3\theta\frac{d}{d\theta}(\sec^2\theta) + \sec^2\theta\cdot\frac{d}{d\theta}(\tan^3\theta)$$

$$\frac{dy}{d\theta} = \tan^3\theta\cdot 2\sec^2\theta\cdot\tan\theta + \sec^2\theta\, 3\tan^2\theta\sec^2\theta$$

$$\frac{dy}{d\theta} = 2\tan^4\theta\sec^2\theta + 3\sec^4\theta\cdot\tan^2\theta$$

$$\frac{dy}{d\theta} = \tan^2\theta\sec^2\theta[2\tan^2\theta + 3\sec^2\theta] \quad \text{Ans.}$$

(iii) Let $y = (\sin 2\theta - \cos 3\theta)^2$

Diff. w.r.t θ, we have

$$\frac{dy}{d\theta} = 2(\sin 2\theta - \cos 3\theta).(2\cos 2\theta + 3\sin 3\theta)$$

$$\frac{dy}{dx} = 2(\sin 2\theta - \cos 3\theta)(2\cos 2\theta + 3\sin 3\theta)$$ Ans.

(iv) Let $y = \cos\sqrt{x} + \sqrt{\sin x}$

Diff. w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}(\cos\sqrt{x}) + \frac{d}{dx}(\sqrt{\sin x})$$

$$\frac{dy}{dx} = -\sin\sqrt{x}.\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{\sin x}}\cdot\cos x$$

$$\frac{dy}{dx} = \frac{-\sin\sqrt{x}}{2\sqrt{x}} + \frac{\cos x}{2\sqrt{\sin x}}$$ Ans.

Q.3 Find $\frac{dy}{dx}$ if

(i) $y = x\cos y$.

Diff. w.r.t x, we have

$$\frac{dy}{dx} = \frac{d}{dx}(x\cdot\cos y)$$

$$\frac{dy}{dx} = x\frac{d}{dx}(\cos y) + \cos y\cdot\frac{dx}{dx}$$

$$\frac{dy}{dx} = x\cdot(-\sin y)\frac{dy}{dx} + \cos y\cdot 1$$

$$\frac{dy}{dx} + x\sin y\frac{dy}{dx} = \cos y$$

$$(1 + x\sin y)\frac{dy}{dx} = \cos y$$

$$\frac{dy}{dx} = \frac{\cos y}{(1 + x\sin y)}$$ Ans.

(ii) $x = y\sin y$

Diff. w.r.t x

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$$\frac{dx}{dx} = \frac{d}{dx}(y\cdot\sin y)$$

$$1 = y\frac{d}{dx}(\sin y) + \sin y\cdot\frac{dy}{dx}$$

$$1 = y\cos y\frac{dy}{dx} + \sin y\frac{dy}{dx}$$

$$1 = (y\cos y + \sin y)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{(y\cos y + \sin y)}$$ Ans

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Q.4 Find derivative w.r.t. x

(i) $y = \cos\sqrt{\frac{1+x}{1+2x}}$ (Let)

Diff w.r.t. x, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left(\cos\sqrt{\frac{1+x}{1+2x}}\right)$$

$$\frac{dy}{dx} = -\sin\sqrt{\frac{1+x}{1+2x}}.\frac{d}{dx}\left(\frac{1+x}{1+2x}\right)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = -\sin\sqrt{\frac{1+x}{1+2x}}\,\frac{1}{2}\left(\frac{1+x}{1+2x}\right)^{-\frac{1}{2}}\frac{d}{dx}\left(\frac{1+x}{1+2x}\right)$$

$$\frac{dy}{dx} = \frac{-1}{2}\sin\sqrt{\frac{1+x}{1+2x}}\left(\frac{1+x}{1+2x}\right)^{-\frac{1}{2}}\cdot\frac{(1+2x)(0+1)-(1+x)(0+2)}{(1+2x)^2}$$

$$\frac{dy}{dx} = \frac{-1}{2}\sin\sqrt{\frac{1+x}{1+2x}}\cdot\left(\frac{1+2x}{1+x}\right)^{\frac{1}{2}}\frac{1+\cancel{2x}-2-\cancel{2x}}{(1+2x)^2}$$

$$\frac{dy}{dx} = \frac{-1}{2}\sin\sqrt{\frac{1+x}{1+2x}}\cdot\frac{-1}{\sqrt{(1+x)}\,(1+2x)^{3/2}}$$

$$\frac{dy}{dx} = \frac{1}{2}\sin\sqrt{\frac{1+x}{1+2x}}\cdot\frac{1}{\sqrt{1+x}\,(1+2x)^{3/2}}$$ Ans.

(ii) $y = \sin\sqrt{\frac{1+2x}{1+x}}$ (Let)

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sin\sqrt{\frac{1+2x}{1+x}}\right)$$

$$\frac{dy}{dx} = \cos\sqrt{\frac{1+2x}{1+x}}\;\frac{d}{dx}\left(\frac{1+2x}{1+x}\right)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \cos\sqrt{\frac{1+2x}{1+x}}\cdot\frac{1}{2}\left(\frac{1+2x}{1+x}\right)^{-1/2}\frac{d}{dx}\left(\frac{1+2x}{1+x}\right)$$

$$\frac{dy}{dx} = \frac{1}{2}\cos\sqrt{\frac{1+2x}{1+x}}\left(\frac{1+x}{1+x2}\right)^{1/2}\cdot\frac{(1+x)(0+2)-(1+2x)(0+1)}{(1+x)^2}$$

$$\frac{dy}{dx} = \frac{1}{2}\cos\sqrt{\frac{1+2x}{1+x}}\cdot\frac{(1+x)^{1/2}}{(1+2x)^{1/2}}\cdot\frac{2+\cancel{2x}-1-\cancel{2x}}{(1+x)^2}$$

$$\frac{dy}{dx} = \frac{1}{2}\cos\sqrt{\frac{1+2x}{1+x}}\cdot\frac{1}{\sqrt{1+2x}\cdot(1+x)^{3/2}} \quad \text{Ans.}$$

Q.5 Differentiate

(i) $\sin x$ w.r.t. $\cot x$

Let $y = \sin x$ and $u = \cot x$

$$\frac{dy}{dx} = \frac{d}{dx}(\sin x) \qquad \frac{du}{dx} = \frac{d}{dx}(\cot x)$$

$$\frac{dy}{dx} = \cos x \qquad \frac{du}{dx} = -\operatorname{cosec}^2 x$$

Now using chain rule

$$\frac{dy}{du} = \frac{dy}{dx}\cdot\frac{dx}{du}$$

$$\frac{dy}{du} = \cos x\cdot\frac{-1}{\operatorname{cosec}^2 x}$$

$$\frac{dy}{du} = -\cos x\,\sin^2 x \quad \text{Ans.}$$

(ii) $\sin^2 x$ w.r.t $\cos^4 x$

Let $y = \sin^2 x$ and $u = \cos^4 x$

$$\frac{dy}{dx} = 2\sin x\cdot\cos x \qquad \frac{du}{dx} = 4\cos^3 x(-\sin x)$$

$$\frac{dy}{dx} = 2\sin x\cos x \qquad \frac{du}{dx} = -4\cos^3 x\sin x$$

Now using chain Rule.

$$\frac{dy}{du} = \frac{dy}{dx}\cdot\frac{dx}{du}$$

$$\frac{dy}{du} = \cancel{2}\cancel{\sin x}\cos x\cdot\frac{-1}{2\cancel{4}\cos^{3-1}x\,\cancel{\sin x}}$$

$$\frac{dy}{du} = \frac{-1}{2\cos^2 x} = \frac{-1}{2}\sec^2 x \quad \text{Ans.}$$

Q.6 If $\tan y(1+\tan x) = 1-\tan x$ then show that $\frac{dy}{dx} = -1$

Sol:- Consider

$$\tan y(1+\tan x) = 1-\tan x$$

$$\tan y = \frac{1-\tan x}{1+1.\tan x}$$

$\because \tan\frac{\pi}{4} = 1$

$$\Rightarrow \tan y = \frac{\tan\pi/4-\tan x}{1+\tan\pi/4.\tan x}$$

$\because \tan(\alpha-\beta) = \dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$

$$y = \tan^{-1}\left[\tan\left(\frac{\pi}{4}-x\right)\right]$$

$$y = \frac{\pi}{4}-x$$

Diff. w.r.t. "x"

$$\frac{dy}{dx} = 0-1$$

$$\frac{dy}{dx} = -1 \quad \text{(Proved)}$$

Q.7 If $y = \sqrt{\tan x+\sqrt{\tan x+\cdots+\infty}}$ then Prove that $(2y-1)\frac{dy}{dx} = \sec^2 x$

Proof:- $y = \sqrt{\tan x+\sqrt{\tan x+\cdots+\infty}}$

Squaring on both sides.

$$y^2 = \tan x+\sqrt{\tan x+\sqrt{\tan x+\cdots\infty}}$$

$$y^2 = \tan x+y$$

Diff w.r.t. x, we have

$$2y\frac{dy}{dx} = \sec^2 x+\frac{dy}{dx}$$

$$2y\frac{dy}{dx}-\frac{dy}{dx} = \sec^2 x$$

$$(2y-1)\frac{dy}{dx} = \sec^2 x \quad \text{(Proved)}.$$

Q.8 If $x = a\cos^3\theta$, $y = b\sin^3\theta$,

Show that $a\dfrac{dy}{dx} + b\tan\theta = 0$

Proof:- $x = a\cos^3\theta$, $y = b\sin^3\theta$

$$\frac{dx}{d\theta} = a \cdot 3\cos^2\theta(-\sin\theta) \qquad \frac{dy}{d\theta} = 3b\sin^2\theta\cos\theta$$

$$\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta \qquad \frac{dy}{d\theta} = 3b\sin^2\theta\cos\theta$$

Now Using Chain Rule

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$$

$$\frac{dy}{dx} = \cancel{3}b\sin^{\cancel{2}}\theta\,\cancel{\cos\theta} \cdot \frac{-1}{\cancel{3}a\cos^{\cancel{2}}\theta\,\cancel{\sin\theta}}$$

$$\frac{dy}{dx} = \frac{-b}{a}\,\frac{\sin\theta}{\cos\theta}$$

$$a\frac{dy}{dx} = -b\tan\theta$$

$$a\frac{dy}{dx} + b\tan\theta = 0 \quad (\text{Proved}).$$

Q.9 $x = a(\cos t + \sin t)$, $y = a(\sin t - t\cos t)$

$$\frac{dx}{dt} = a(-\sin t + \cos t) = a(\cos t - \sin t)$$

Now $\dfrac{dy}{dt} = a(\cos t - 1\cdot\cos t - t\cdot(-\sin t))$

$$\frac{dy}{dt} = a(\cancel{\cos t} - \cancel{\cos t} + t\sin t)$$

$$\frac{dy}{dt} = a(t\sin t)$$

Now Using Chain Rule.

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$\frac{dy}{dx} = \cancel{a}(t\sin t) \cdot \frac{1}{\cancel{a}(\cos t - \sin t)}$$

$$\frac{dy}{dx} = \frac{t\sin t}{\cos t - \sin t} \quad \text{Ans.}$$

Q.10 Differentiate w.r.t. "x"

(i) $y = \cos^{-1}\left(\dfrac{x}{a}\right)$ (Let)

Diff. w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\cos^{-1}\frac{x}{a}\right)$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-\left(\frac{x}{a}\right)^2}} \cdot \frac{d}{dx}\left(\frac{x}{a}\right)$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{\frac{a^2-x^2}{a^2}}} \cdot \frac{1}{a}$$

$$\frac{dy}{dx} = \frac{-\cancel{a}}{\sqrt{a^2-x^2}} \cdot \frac{1}{\cancel{a}} = \frac{-1}{\sqrt{a^2-x^2}} \quad \text{Ans.}$$

(ii) $y = \cot^{-1}\left(\dfrac{x}{a}\right)$ (Let)

Diff. w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\cot^{-1}\frac{x}{a}\right)$$

$$\frac{dy}{dx} = \frac{-1}{1+\left(\frac{x}{a}\right)^2} \cdot \frac{d}{dx}\left(\frac{x}{a}\right)$$

$$\frac{dy}{dx} = \frac{-1}{\frac{a^2+x^2}{a^2}} \cdot \frac{1}{a} = \frac{-a^2}{a^2+x^2} \cdot \frac{1}{a}$$

$$\frac{dy}{dx} = \frac{-a}{a^2+x^2} \quad \text{Ans.}$$

(iii) $y = \dfrac{1}{a}\sin^{-1}\left(\dfrac{a}{x}\right)$ (Let)

Diff. w.r.t. x

$$\frac{dy}{dx} = \frac{1}{a}\,\frac{d}{dx}\left(\sin^{-1}\left(\frac{a}{x}\right)\right)$$

$$\frac{dy}{dx} = \frac{1}{a} \cdot \frac{1}{\sqrt{1-\left(\frac{a}{x}\right)^2}} \cdot \frac{d}{dx}\left(\frac{a}{x}\right)$$

$$\frac{dy}{dx} = \frac{1}{a}\,\frac{1}{\sqrt{\frac{x^2-a^2}{x^2}}} \cdot \frac{x\cdot 0 - a\cdot 1}{x^2}$$

$$\frac{dy}{dx} = \frac{x}{a\sqrt{x^2-a^2}} \cdot \frac{-a}{x^2}$$

$$\frac{dy}{dx} = \frac{-1}{x\sqrt{x^2-a^2}} \quad \text{Ans.}$$

(iv) $y = \sin^{-1}\sqrt{1-x^2}$ (Let)

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sin^{-1}\sqrt{1-x^2}\right)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(\sqrt{1-x^2})^2}}\cdot\frac{d}{dx}\left[\sqrt{1-x^2}\right]$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(1-x^2)}}\cdot\frac{1}{2\sqrt{1-x^2}}\cdot(0-2x)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{\cancel{1}-\cancel{1}+x^2}}\cdot\frac{-\cancel{2}x}{\cancel{2}\sqrt{1-x^2}}$$

$$\frac{dy}{dx} = \frac{-x}{x\sqrt{1-x^2}} = \frac{-1}{\sqrt{1-x^2}} \text{ Ans.}$$

(v) $y = \sec^{-1}\left[\frac{x^2+1}{x^2-1}\right]$ (Let)

Diff. w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sec^{-1}\left(\frac{x^2+1}{x^2-1}\right)\right)$$

$$\frac{dy}{dx} = \frac{1}{\left(\frac{x^2+1}{x^2-1}\right)\sqrt{\left(\frac{x^2+1}{x^2-1}\right)^2-1}}\cdot\frac{d}{dx}\left(\frac{x^2+1}{x^2-1}\right)$$

$$\frac{dy}{dx} = \frac{1}{\left(\frac{x^2+1}{x^2-1}\right)\sqrt{\frac{(x^2+1)^2-(x^2-1)^2}{(x^2-1)^2}}}\cdot\frac{(x^2-1)(2x+0)-(x^2+1)(2x-0)}{(x^2-1)^2}$$

$$\frac{dy}{dx} = \frac{1}{\frac{(x^2+1)}{(x^2-1)^2}\sqrt{\cancel{x^4}+\cancel{1}+2x^2-\cancel{x^4}-\cancel{1}+2x^2}}\cdot\frac{\cancel{2x^3}-2x-\cancel{2x^3}-2x}{(x^2-1)^2}$$

$$\frac{dy}{dx} = \frac{\cancel{(x^2-1)^2}}{(x^2+1)\sqrt{4x^2}}\cdot\frac{-4x}{\cancel{(x^2-1)^2}}$$

$$\frac{dy}{dx} = \frac{-\cancel{4}x^2}{\cancel{2x}(x^2+1)} = \frac{-2}{x^2+1} \text{ Ans.}$$

(vi) $y = \cot^{-1}\left(\frac{2x}{1-x^2}\right)$ (Let)

Diff. w.r.t "x"

$$\frac{dy}{dx} = \frac{d}{dx}\left[\cot^{-1}\left(\frac{2x}{1-x^2}\right)\right]$$

$$\frac{dy}{dx} = \frac{-1}{1+\left(\frac{2x}{1-x^2}\right)^2}\cdot\frac{d}{dx}\left(\frac{2x}{1-x^2}\right)$$

$$\frac{dy}{dx} = \frac{-1}{\frac{(1-x^2)^2+(2x)^2}{(1-x^2)^2}}\cdot\frac{(1-x^2)(2\cdot1)-2x(0-2x)}{(1-x^2)^2}$$

$$\frac{dy}{dx} = \frac{-\cancel{(1-x^2)^2}}{1+x^4-2x^2+4x^2}\cdot\frac{2-2x^2+4x^2}{\cancel{(1-x^2)^2}}$$

$$\frac{dy}{dx} = \frac{-(2+2x^2)}{(1+x^4+2x^2)}$$

$$\frac{dy}{dx} = \frac{-2(1+x^2)}{(1+x^2)^2} = \frac{-2}{(1+x^2)} \text{ Ans.}$$

(vii) $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ (Let)

Diff. w.r.t. "x", we have

$$\frac{dy}{dx} = \frac{d}{dx}\left(\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right)$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-\left(\frac{1-x^2}{1+x^2}\right)^2}}\cdot\frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{\frac{(1+x^2)^2-(1-x^2)^2}{(1+x^2)^2}}}\cdot\frac{(1+x^2)(0-2x)-(1-x^2)(0+2x)}{(1+x^2)^2}$$

$$\frac{dy}{dx} = \frac{-(1+x^2)}{\sqrt{\cancel{1}+\cancel{x^4}+2x^2-\cancel{1}-\cancel{x^4}+2x^2}}\cdot\frac{-2x-\cancel{2x^3}-2x+\cancel{2x^3}}{(1+x^2)^2}$$

$$\frac{dy}{dx} = \frac{-\cancel{(1+x^2)}}{\sqrt{4x^2}}\cdot\frac{-4x}{(1+x^2)^{\cancel{2}}}$$

$$\frac{dy}{dx} = \frac{2\cancel{4}x}{\cancel{2x}(1+x^2)}$$

$$\frac{dy}{dx} = \frac{2}{(1+x^2)} \text{ Ans.}$$

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Q.11 If $\frac{y}{x}=\tan^{-1}\left(\frac{x}{y}\right)$ then show $\frac{dy}{dx}=\frac{y}{x}$ **(25)**

Proof: $\frac{y}{x}=\tan^{-1}\left(\frac{x}{y}\right)$

$y = x\cdot\tan^{-1}\left(\frac{x}{y}\right)$

Diff. w.r.t. 'x'

$\frac{dy}{dx} = x\cdot\frac{d}{dx}\left(\tan^{-1}\frac{x}{y}\right)+\tan^{-1}\left(\frac{x}{y}\right)\cdot\frac{d}{dx}(x)$

$\frac{dy}{dx} = x\cdot\frac{1}{1+\left(\frac{x}{y}\right)^2}\cdot\frac{d}{dx}\left(\frac{x}{y}\right)+\tan^{-1}\left(\frac{x}{y}\right)\cdot 1$ $\left(\because \tan^{-1}\left(\frac{x}{y}\right)=y/x\right)$

$\frac{dy}{dx} = x\cdot\frac{\cancel{y^2}}{y^2+x^2}\cdot\frac{y\cdot 1 - x\cdot\frac{dy}{dx}}{\cancel{y^2}}+\frac{y}{x}\cdot 1$

$\frac{dy}{dx} = \frac{xy}{x^2+y^2}-\frac{x^2}{x^2+y^2}\frac{dy}{dx}+\frac{y}{x}$

$\frac{dy}{dx}+\frac{x^2}{x^2+y^2}\frac{dy}{dx} = \frac{xy}{x^2+y^2}+\frac{y}{x}$

$\left(1+\frac{x^2}{x^2+y^2}\right)\frac{dy}{dx} = \frac{x^2y+y(x^2+y^2)}{x(x^2+y^2)}$

$\left(\frac{x^2+y^2+x^2}{x^2+y^2}\right)\frac{dy}{dx} = \frac{y(x^2+x^2+y^2)}{x(x^2+y^2)}$

$\left(\frac{2x^2+y^2}{x^2+y^2}\right)\frac{dy}{dx} = \frac{y}{x}\left(\frac{2x^2+y^2}{x^2+y^2}\right)$

$\frac{dy}{dx} = \frac{y}{x}\left(\frac{\cancel{2x^2+y^2}}{\cancel{x^2+y^2}}\right)\cdot\left(\frac{\cancel{x^2+y^2}}{\cancel{2x^2+y^2}}\right)$

$\frac{dy}{dx} = y/x$. (Proved).

Q.12 If $y=\tan(p\tan^{-1}x)$,

Show $(1+x^2)y_1 - p(1+y^2)=0$

Proof: $y=\tan(p\tan^{-1}x)$

$\tan^{-1}y = p\tan^{-1}x$

Diff. w.r.t. x

$\frac{1}{1+y^2}\frac{dy}{dx} = p\cdot\frac{1}{1+x^2}\cdot 1$ $\left(\because \frac{dy}{dx}=y_1\right)$

$\frac{1}{1+y^2}\cdot y_1 = p\cdot\frac{1}{1+x^2}$

$(1+x^2)y_1 = p(1+y^2)$

$(1+x^2)y_1 - p(1+y^2)=0$ (Proved).

The useful derivatives

(1) $\frac{d}{dx}(a^x) = a^x\cdot\ln a$ $a>0$ $a\neq 1$ $a\in Z^+$

(2) $\frac{d}{dx}(e^x) = e^x\cdot 1$ $e\approx 2.7183$

(3) $\frac{d}{dx}(\log_a x) = \frac{1}{x}\cdot\frac{1}{\ln a}$ $a>0$ $a\neq 1$

(4) $\frac{d}{dx}(\ln x) = \frac{1}{x}\cdot 1$

Derivatives of Hyperbolic functions.

(1) $\frac{d}{dx}(\sinh x) = \cosh x\cdot 1$

(2) $\frac{d}{dx}(\cosh x) = \sinh x\cdot 1$

(3) $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x\cdot 1$

(4) $\frac{d}{dx}(\coth x) = -\operatorname{cosech}^2 x\cdot 1$

(5) $\frac{d}{dx}(\operatorname{sech} x) = -\tanh x\cdot\operatorname{sech} x\cdot 1$

(6) $\frac{d}{dx}(\operatorname{cosech} x) = -\coth x\cdot\operatorname{cosech} x\cdot 1$

Derivatives of Inverse Hyperbolic functions.

(1) $\frac{d}{dx}(\sinh^{-1}x) = \frac{1}{\sqrt{1+x^2}}\cdot 1$

(2) $\frac{d}{dx}(\cosh^{-1}x) = \frac{1}{\sqrt{x^2-1}}\cdot 1$

(3) $\frac{d}{dx}(\tanh^{-1}x) = \frac{1}{1-x^2}\cdot 1$

(4) $\frac{d}{dx}(\coth^{-1}x) = \frac{1}{1-x^2}\cdot 1$

(5) $\frac{d}{dx}(\operatorname{sech}^{-1}x) = \frac{-1}{x\sqrt{1-x^2}}\cdot 1$

(6) $\frac{d}{dx}(\operatorname{cosech}^{-1}x) = \frac{-1}{x\sqrt{1+x^2}}\cdot 1$