Answers 2.3

Q.1 (i) $\frac{-y}{x+2y}$ (ii) $-\frac{b^2x}{a^2y}$ (iii) $-\frac{y^{1/3}}{x^{1/3}}$

(iv) $\frac{-\sqrt{1+y}\left[y+2\sqrt{(1+x)(1+y)}\right]}{\sqrt{1+x}\left[x+2\sqrt{(1+x)(1+y)}\right]}$ (v) $\frac{3x^2-4y^2}{8xy+4y+15y^2}$

(vi) $-\left(\frac{y}{x}\right)^{3/2}$ (vii) $x/3y$ (viii) $-x^2/y^2$

(ix) $\frac{x-y}{x-2y}$ (x) $-\frac{x^2+2y}{2x+5y^2}$

Q.2 (i) $\frac{2a}{y}$ (ii) $\frac{3x^2-y}{x-3y^2}$ (iii) $\frac{x(2a^2y^2-x^3)}{y(y^3-2a^2x^2)}$

(iv) $\frac{1}{(x+1)^{3/2}\sqrt{x-1}}$

Q.3 (i) $4t$ (ii) $\frac{4\theta+1}{2\theta-1}$ (iii) $\frac{b}{3a}(\theta^{-2}+\theta^{-4})$

(iv) $\frac{b(t^2-1)}{2at}$ (v) $\frac{t(2-t^3)}{1-2t^3}$ (vi) $\frac{b}{a}t$

(vii) $\frac{t^2-1}{2t}$

2.7 Differentiation of Function w.r.t. Another Function

Sometime we differentiate given function w.r.t. another function instead of independent variable.

In this case we proceed as follows:

Example 1:

Differentiate x^3+8 w.r.t. x^2+4

Solution:

Let $y = x^3+8$ $u = x^2+4$

$$\frac{dy}{dx} = 3x^2 \qquad \frac{du}{dx} = 2x$$

Using chain rule

$$\frac{dy}{du} = \frac{dy}{dx} \times \frac{dx}{du}$$

$$\frac{dy}{du} = 3x^2 \cdot \frac{1}{2x} = \frac{3x}{2}$$

Example 2:

Differentiate $\frac{x^2}{1+x^2}$ w.r.t. x^4

Solution:

Let $y = \frac{x^2}{1+x^2}$ and $u = x^4$

$$\frac{dy}{dx} = \frac{(1+x^2)(2x) - x^2(2x)}{(1+x^2)^2} \qquad \frac{du}{dx} = 4x^3$$

$$= \frac{2x}{(1+x^2)^2}$$

By chain Rule $\frac{dy}{du} = \frac{dy}{dx} \cdot \frac{dy}{du}$

$$\frac{dy}{du} = \frac{2x}{(1+x^2)^2} \times \frac{1}{4x^3}$$

$$\frac{dy}{du} = \frac{1}{2x^2(1+x^2)^2}$$

2.8 Differentiation by Rationalization

In order to differentiate irrational algebraic expression, it is useful to rationalize the given expression before differentiation on.

Example 3:

Differentiate w.r.t. x

$$y = \frac{\sqrt{a+x} - \sqrt{a-x}}{\sqrt{a+x} + \sqrt{a-x}}$$

Solution:

$$y = \frac{\sqrt{a+x}-\sqrt{a-x}}{\sqrt{a+x}+\sqrt{a-x}} \times \frac{\sqrt{a+x}-\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}} \quad \text{(By rationalization)}$$

$$= \frac{a+x+a-x-2\sqrt{(a+x)(a-x)}}{a+x-a+x}$$

$$y = \frac{a-\sqrt{a^2-x^2}}{x}$$

$$\therefore \quad \frac{dy}{du} = \frac{x\frac{d}{dx}(a-\sqrt{a^2-x^2})-(a-\sqrt{a^2-x^2})\frac{d}{dx}(x)}{x^2}$$

$$= \frac{x\left\{-\frac{1}{2}(a^2-x^2)^{-1/2}(-2x)\right\}-(a-\sqrt{a^2-x^2})(1)}{x^2}$$

$$= \frac{\frac{x^2}{\sqrt{a^2-x^2}}+(\sqrt{a^2-x^2}-a)}{x^2}$$

$$= \frac{x^2+a^2-x^2-a\sqrt{a^2-x^2}}{x^2\sqrt{a^2-x^2}}$$

$$\frac{dy}{du} = \frac{a(a-\sqrt{a^2-x^2})}{x^2\sqrt{a^2-x^2}}$$

Example 4:

Differentiate $y = \frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}$

Solution:

$$y = \frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}} \times \frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}$$

$$y = \frac{\left(\sqrt{1+x^2}+\sqrt{1-x^2}\right)^2}{1+x^2-1+x^2}$$

$$= \frac{1+x^2+1-x^2+2\sqrt{(1+x^2)(1-x^2)}}{2x^2} = \frac{2+2\sqrt{1-x^4}}{2x^2} = \frac{1+\sqrt{1-x^4}}{x^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{x^2 \frac{d}{dx}(1+\sqrt{1-x^4}) - (1+\sqrt{1-x^4})\frac{d}{dx}(x^2)}{(x^2)^2}$$

$$= \frac{x^2\left\{\frac{-4x^3}{2\sqrt{1-x^4}}\right\} - 2x\left(\frac{1+\sqrt{1-x^4}}{1}\right)}{x^4}$$

$$= \frac{-2x^5 - 2x\sqrt{1-x^4} - 2x(1-x^4)}{x^4\sqrt{1-x^4}}$$

$$= \frac{-2x^5 - 2x\sqrt{1-x^4} - 2x + 2x^5}{(x^2)^2}$$

$$\frac{dy}{dx} = \frac{-2x(1+\sqrt{1-x^4})}{x^4\sqrt{1-x^4}}$$

Exercise 2.4

Q.1. Differentiate the following:

i. $\frac{x^2}{x^2-1}$ w.r.t. x^2

ii. $\frac{x^2+1}{x^2-1}$ w.r.t. $\frac{x-1}{x+1}$

iii. $2x^2+x+1$ w.r.t. x^2-x-1

iv. $x^2-\frac{1}{x^2}$ w.r.t. x^4

v. $\frac{ax+b}{cx+d}$ w.r.t. $\frac{ax^2+b}{cx^2+d}$

vi. $\frac{ax+b}{cx+d}$ w.r.t. $\frac{px+q}{rx+s}$

vii. $\frac{x^2+a^2}{x^2-a^2}$ w.r.t. $\frac{x-a}{x+a}$

Q.2: Find the derivative of the following:

$$\frac{\sqrt{x^2+1}-\sqrt{x^2-1}}{\sqrt{x^2-1}+\sqrt{x^2-1}}$$

Answers 2.4

Q.1. i. $\frac{-1}{(x^2 - 1)^2}$ ii. $\frac{-2x}{(x - 1)^2}$ iii. $\frac{4x + 1}{2x - 1}$

iv. $\frac{x^4 + 1}{2x^6}$ v. $\frac{(cx^2 + d)^2}{2x(cx + d)^2}$

vi. $\frac{ad - bc}{ps - rq} \times \left(\frac{rx + s}{cx + d}\right)^2$ vii. $\frac{-2ax}{(x - a)^2}$

Q.2 $2x - \frac{2x^3}{\sqrt{x^4 - 1}}$