Dividing by δz where $\lim_{\delta z \to 0}$

$$\lim_{\delta z \to 0} \frac{\delta y}{\delta z} = \lim_{\delta z \to 0} \frac{\delta z\left[-7(az-b)^{-8} + 28\delta z(az-b)^{-9} + \cdots\right]}{\delta z}$$

$$\frac{dy}{dz} = \lim_{\delta z \to 0} -7(az-b)^{-8} + 28\delta z(az-b)^{-9} + \cdots$$

By Applying Limit

$$\frac{dy}{dx} = -7(az-b)^{-8} + 0 + \cdots$$

$$\boxed{\frac{dy}{dz} = \frac{-7}{(az-b)^8}}$$

(Similarly other parts do yourself.)

Theorems of Derivative:-

(1) $\frac{d}{dx}[\text{Constant}] = 0$

(2) $\frac{d}{dx}[x] = 1$

(3) $\frac{d}{dx}[cx] = c\,\frac{d(x)}{dx}$

(4) $\frac{d}{dx}[x^n] = n x^{n-1} \cdot 1$ (Power Rule)

(5) $\frac{d}{dx}[u.v] = \frac{du}{dx}.v + u\,\frac{dv}{dx}$ (Product Rule)

(6) $\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{v\frac{du}{dx} - u.\frac{dv}{dx}}{v^2}$ (Quotient Rule)

(7) $\frac{d}{dx}[u \pm v] = \frac{du}{dx} \pm \frac{dv}{dx}$

(8) Chain Rule

If $x = u(t)$ $\quad y = v(t)$

then $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$.

Exercise 2.3

Differentiate w.r.t. 'x'

(1) $x^4 + 2x^3 + x^2$

Let $y = x^4 + 2x^3 + x^2$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = \frac{d}{dx}(x^4) + 2\frac{d}{dx}(x^3) + \frac{d}{dx}(x^2)$$

$$\frac{dy}{dx} = 4x^3 + 2.3x^2 + 2x$$

$$\frac{dy}{dx} = 4x^3 + 6x^2 + 2x \quad \text{Ans.}$$

(2) $x^{-3} + 2x^{-3/2} + 3$

Let $y = x^{-3} + 2x^{-3/2} + 3$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = \frac{d}{dx}(x^{-3}) + 2\frac{d}{dx}(x^{-3/2}) + \frac{d}{dx}(3)$$

$$\frac{dy}{dx} = -3x^{-4} + 2\left(\frac{-3}{2}\right) \cdot x^{-5/2} + 0$$

$$\frac{dy}{dx} = \frac{-3}{x^4} - \frac{3}{x^{5/2}} = -3\left[\frac{1}{x^4} + \frac{1}{x^{5/2}}\right] \quad \text{Ans.}$$

(3) $\frac{a+x}{a-x}$

Let $y = \frac{a+x}{a-x}$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{a+x}{a-x}\right]$$

$$\frac{dy}{dx} = \frac{(a-x)\frac{d}{dx}(a+x) - (a+x)\frac{d}{dx}(a-x)}{(a-x)^2}$$

$$\frac{dy}{dx} = \frac{(a-x)(0+1) - (a+x)(0-1)}{(a-x)^2}$$

$$\frac{dy}{dx} = \frac{a - x + a + x}{(a-x)^2} = \frac{2a}{(a-x)^2}$$

$$\frac{dy}{dx} = \frac{2a}{(a-x)^2} \quad \text{Ans}$$

(4) $\frac{2x-3}{2x+1}$

Let $y = \frac{2x-3}{2x+1}$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{2x-3}{2x+1}\right)$$

$$\frac{dy}{dx} = \frac{(2x+1)\frac{d}{dx}(2x-3) - (2x-3)\frac{d}{dx}(2x+1)}{(2x+1)^2}$$

$$\frac{dy}{dx} = \frac{(2x+1)(2.1-0) - (2x-3)(2.1+0)}{(2x+1)^2}$$

$$\frac{dy}{dx} = \frac{2(2x+1) - 2(2x-3)}{(2x+1)^2} = \frac{4x+2-4x+6}{(2x+1)^2}$$

$$dy/dx = \frac{8}{(2x+1)^2} \quad \text{Ans.}$$

(5) $(x-5)(3-x)$

Let $y = (x-5)(3-x) = 3x - 15 - x^2 + 5x$

$$y = 8x - x^2 - 15$$

Diff. w.r.t. "x"

$$\frac{dy}{dx} = 8\frac{d}{dx}(x) - \frac{d}{dx}(x^2) + \frac{d}{dx}(15)$$

$$\frac{dy}{dx} = 8\cdot 1 - 2x + 0 = 8 - 2x \quad \text{Ans.}$$

(6) $\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2$

Let $y = \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 = \left(x + \frac{1}{x} - 2\right)$

$$y = (x + x^{-1} - 2)$$

diff. w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}(x) + \frac{d}{dx}(x^{-1}) - \frac{d}{dx}(2)$$

$$= 1 - x^{-2}\cdot 1 - 0$$

$$\frac{dy}{dx} = 1 - \frac{1}{x^2} \quad \text{Ans.}$$

(7). Let $y = \frac{(1+\sqrt{x})(x - x^{3/2})}{\sqrt{x}}$

$$y = \frac{(1+\sqrt{x})(1-\sqrt{x})\,x}{\sqrt{x}} = [1-(\sqrt{x})^2]\sqrt{x}$$

$$y = (1-x)\sqrt{x} = \sqrt{x} - x^{3/2}$$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = \frac{d}{dx}(x^{1/2}) - \frac{d}{dx}(x^{3/2})$$

$$dy/dx = \frac{1}{2}x^{-1/2}\cdot 1 - \frac{3}{2}x^{1/2}\cdot 1$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} - \frac{3}{2}\sqrt{x} \quad \text{Ans.}$$

(8) Let $y = \frac{(x^2+1)^2}{x^2-1}$

Diff. w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{(x^2+1)^2}{x^2-1}\right]$$

$$\frac{dy}{dx} = \frac{(x^2-1)\frac{d}{dx}(x^2+1)^2 - (x^2+1)^2\frac{d}{dx}(x^2-1)}{(x^2-1)^2}$$

$$\frac{dy}{dx} = \frac{(x^2-1)\cdot 2(x^2+1)^1\cdot 2x - (x^2+1)^2\cdot 2x}{(x^2-1)^2}$$

$$dy/dx = \frac{4x(x^4-1) - 2x(x^4+2x^2+1)}{(x^2-1)^2}$$

$$dy/dx = \frac{4x^5 - 4x - 2x^5 - 4x^3 - 2x}{(x^2-1)^2}$$

$$\frac{dy}{dx} = \frac{2x^5 - 4x^3 - 6x}{(x^2-1)^2} \quad \text{Ans.}$$

(9). Let $y = \frac{x^2+1}{x^2-3}$

Diff. w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^2+1}{x^2-3}\right)$$

$$\frac{dy}{dx} = \frac{(x^2-3)\cdot\frac{d}{dx}(x^2+1) - (x^2+1)\cdot\frac{d}{dx}(x^2-3)}{(x^2-3)^2}$$

$$\frac{dy}{dx} = \frac{(x^2-3)(2x+0) - (x^2+1)(2x-0)}{(x^2-3)^2}$$

$$\frac{dy}{dx} = \frac{2x^3 - 6x - 2x^3 - 2x}{(x^2-3)^2}$$

$$\frac{dy}{dx} = \frac{-8x}{(x^2-3)^2} \quad \text{Ans.}$$

10) Let $y = \frac{\sqrt{1+x}}{\sqrt{1-x}} = \left(\frac{1+x}{1-x}\right)^{1/2}$

Diff. w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1+x}{1-x}\right)^{1/2}$$

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{1+x}{1-x}\right)^{-1/2}\cdot\frac{d}{dx}\left(\frac{1+x}{1-x}\right)$$

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{1-x}{1+x}\right)^{1/2}\cdot\frac{(1-x)\cdot\frac{d}{dx}(1+x) - (1+x)\frac{d}{dx}(1-x)}{(1-x)^2}$$

$$\frac{dy}{dx} = \frac{1}{2}\frac{(1-x)^{1/2}}{(1+x)^{1/2}}\cdot\frac{(1-x)(0+1) - (1+x)(0-1)}{(1-x)^2}$$

$$dy/dx = \frac{1-x+1+x}{2(1+x)^{1/2}\cdot(1-x)^{2-1/2}}$$

$$dy/dx = \frac{2}{2\sqrt{1+x}\,(1-x)^{3/2}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+x}\,(1-x)^{3/2}} \quad \text{Ans.}$$

11) Let $y = \frac{2x-1}{\sqrt{x^2+1}}$

Diff. w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{2x-1}{\sqrt{x^2+1}}\right)$$

$$\frac{dy}{dx} = \frac{\sqrt{x^2+1}\,\frac{d}{dx}(2x-1) - (2x-1)\,\frac{d}{dx}(\sqrt{x^2+1})}{(\sqrt{x^2+1})^2}$$

$$\frac{dy}{dx} = \frac{\sqrt{x^2-1}\cdot(2-0) - (2x-1)\frac{2x}{2\sqrt{x^2+1}}}{(x^2+1)}$$

$$\frac{dy}{dx} = \frac{2\sqrt{x^2+1} - \frac{(2x^2-x)}{\sqrt{x^2+1}}}{(x^2+1)}$$

$$\frac{dy}{dx} = \frac{2(\sqrt{x^2+1})^2 - 2x^2 + x}{(x^2+1)\sqrt{x^2+1}}$$

$$\frac{dy}{dx} = \frac{2(x^2+1) - 2x^2 + x}{(x^2+1)^{3/2}}$$

$$\frac{dy}{dx} = \frac{2x^2 + 2 - 2x^2 + x}{(x^2+1)^{3/2}}$$

$$\frac{dy}{dx} = \frac{2+x}{(x^2+1)^{3/2}} \quad \text{Ans.}$$

12) Let $y = \sqrt{\frac{a-x}{a+x}}$

Diff. w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{a-x}{a+x}\right)^{1/2}$$

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{a-x}{a+x}\right)^{-1/2}\frac{d}{dx}\left(\frac{a-x}{a+x}\right)$$

$$dy/dx = \frac{1}{2}\left(\frac{a+x}{a-x}\right)^{1/2}\cdot\frac{(a+x)\frac{d}{dx}(a-x) - (a-x)\frac{d}{dx}(a+x)}{(a+x)^2}$$

$$dy/dx = \frac{1}{2}\frac{(a+x)^{1/2}}{(a-x)^{1/2}}\cdot\frac{(a+x)(0-1) - (a-x)(0+1)}{(a+x)^2}$$

$$dy/dx = \frac{1}{2}\frac{(a+x)^{1/2}}{\sqrt{a-x}}\cdot\frac{-a-x-a+x}{(a+x)^2}$$

$$dy/dx = \frac{-2a}{2\sqrt{a-x}\cdot(a+x)^{2-1/2}}$$

$$dy/dx = \frac{-a}{\sqrt{a-x}\,(a+x)^{3/2}} \quad \text{Ans.}$$

13. Let $y = \dfrac{\sqrt{x^2+1}}{\sqrt{x^2-1}} = \sqrt{\dfrac{x^2+1}{x^2-1}}$

Diff. w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^2+1}{x^2-1}\right)^{1/2}$$

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{x^2+1}{x^2-1}\right)^{-1/2} \frac{d}{dx}\left(\frac{x^2+1}{x^2-1}\right)$$

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{x^2-1}{x^2+1}\right)^{1/2} \frac{(x^2-1)\frac{d}{dx}(x^2+1)-(x^2+1)\frac{d}{dx}(x^2-1)}{(x^2-1)^2}$$

$$\frac{dy}{dx} = \frac{1}{2}\frac{(x^2-1)^{1/2}}{(x^2+1)^{1/2}} \cdot \frac{(x^2-1)(2x+0)-(x^2+1)(2x-0)}{(x^2-1)^2}$$

$$\frac{dy}{dx} = \frac{(x^2-1)^{1/2}}{2(x^2+1)^{1/2}} \cdot \frac{2x^3-2x-2x^3-2x}{(x^2-1)^2}$$

$$\frac{dy}{dx} = \frac{-4x}{2\sqrt{x^2+1}\cdot(x^2-1)^{2-1/2}}$$

$$dy/dx = \frac{-2x}{\sqrt{x^2+1}\,(x^2-1)^{3/2}} \quad \text{Ans.}$$

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14. Let $y = \dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}$

By Rationalizing, we have

$$y = \frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}} \times \frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}$$

$$y = \frac{(\sqrt{1+x}-\sqrt{1-x})^2}{(\sqrt{1+x})^2-(\sqrt{1-x})^2}$$

$$y = \frac{(\sqrt{1+x})^2+(\sqrt{1-x})^2-2\sqrt{1+x}\sqrt{1-x}}{(1+x)-(1-x)}$$

$$y = \frac{1+x+1-x-2\sqrt{1-x^2}}{1+x-1+x}$$

$$y = \frac{2-2\sqrt{1-x^2}}{2x} = \frac{2(1-\sqrt{1-x^2})}{2x}$$

$$y = \frac{1-\sqrt{1-x^2}}{x}$$

Diff. w.r.t. x, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{1-\sqrt{1-x^2}}{x}\right]$$

$$\frac{dy}{dx} = \frac{x\cdot\frac{d}{dx}(1-\sqrt{1-x^2})-(1-\sqrt{1-x^2})\frac{d}{dx}(x)}{x^2}$$

$$\frac{dy}{dx} = \frac{x\left[0-\frac{-2x}{2\sqrt{1-x^2}}\right]-(1-\sqrt{1-x^2})\cdot 1}{x^2}$$

$$\frac{dy}{dx} = \frac{x\left(\frac{x}{\sqrt{1-x^2}}\right)-1+\sqrt{1-x^2}}{x^2}$$

$$\frac{dy}{dx} = \frac{x^2-\sqrt{1-x^2}+(\sqrt{1-x^2})^2}{x^2\sqrt{1-x^2}} \quad \text{(By L.C.M)}$$

$$\frac{dy}{dx} = \frac{x^2-\sqrt{1-x^2}+1-x^2}{x^2\sqrt{1-x^2}}$$

$$\frac{dy}{dx} = \frac{1-\sqrt{1-x^2}}{x^2\sqrt{1-x^2}} \quad \text{Ans.}$$

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15. Let $y = x\cdot\sqrt{\dfrac{a+x}{a-x}}$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}(x)\sqrt{\frac{a+x}{a-x}} + x\frac{d}{dx}\left(\frac{a+x}{a-x}\right)^{1/2}$$

$$\frac{dy}{dx} = 1\cdot\sqrt{\frac{a+x}{a-x}} + x\frac{1}{2}\left(\frac{a+x}{a-x}\right)^{-1/2}\frac{d}{dx}\left(\frac{a+x}{a-x}\right)$$

$$\frac{dy}{dx} = \sqrt{\frac{a+x}{a-x}} + \frac{x}{2}\sqrt{\frac{a-x}{a+x}}\cdot\frac{(a-x)(0+1)-(a+x)(0-1)}{(a-x)^2}$$

$$\frac{dy}{dx} = \sqrt{\frac{a+x}{a-x}} + \frac{x}{2}\sqrt{\frac{a-x}{a+x}}\;\frac{a-x+a+x}{(a-x)^2}$$

$$\frac{dy}{dx} = \sqrt{\frac{a+x}{a-x}} + \frac{x}{2}\frac{(a-x)^{1/2}}{\sqrt{a+x}}\cdot\frac{2a}{(a-x)^2}$$

$$dy/dx = \sqrt{\frac{a+x}{a-x}} + \frac{x}{\sqrt{a+x}\,(a-x)^{3/2}}$$

---

16. If $y=\sqrt{x}-\frac{1}{\sqrt{x}}$ then prove that

$$2x\frac{dy}{dx}+y=2\sqrt{x}$$

Proof: $y=\sqrt{x}-\frac{1}{\sqrt{x}}$

Diff. w.r.t. x

$$\frac{dy}{dx}=\frac{d}{dx}\left[\sqrt{x}\right]-\frac{d}{dx}\left[x^{-1/2}\right]$$

$$\frac{dy}{dx}=\frac{1}{2\sqrt{x}}+\frac{1}{2}x^{-3/2}=\frac{1}{2\sqrt{x}}+\frac{1}{2x^{3/2}}$$

$$\frac{dy}{dx}=\frac{1}{2\sqrt{x}}\left[1+\frac{1}{x}\right]$$

Multiplying both sides by $2x$

$$2x\frac{dy}{dx}=\frac{2x}{2\sqrt{x}}\left[\frac{x+1}{x}\right]$$

$$2x\frac{dy}{dx}=\frac{x}{\sqrt{x}}+\frac{1}{\sqrt{x}}=\sqrt{x}+\frac{1}{\sqrt{x}}$$

$$2x\frac{dy}{dx}=\sqrt{x}+\frac{1}{\sqrt{x}}$$

Adding y in both sides

$$2x\frac{dy}{dx}+y=\sqrt{x}+\frac{1}{\sqrt{x}}+y$$

$\because y=\sqrt{x}-\frac{1}{\sqrt{x}}$

$$2x\frac{dy}{dx}+y=\sqrt{x}+\frac{1}{\sqrt{x}}+\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)$$

$$2x\frac{dy}{dx}+y=2\sqrt{x}\quad\text{(Proved)}.$$

Q.17 If $y=x^4+2x^2+2$, prove that

$$\frac{dy}{dx}=4x\sqrt{y-1}$$

$\because y=x^4+2x^2+2$

Diff w.r.t. x

$$4x^3+4x+0=4x(x^2+1)$$

$$4x\sqrt{(x^2+1)^2}\qquad(\because a=\sqrt{a^2})$$

$$4x\sqrt{x^4+2x^2+1}$$

$$4x\sqrt{(x^4+2x^2+1)+1-1}$$

$$\frac{dy}{dx}=4x\sqrt{x^4+2x^2+2-1}$$

$$\frac{dy}{dx}=4x\sqrt{y-1}\quad\text{(Proved)}$$

Implicit function:-

The function in which x, y are not expressed in terms of dependent and independent variable are called implicit functions.

Such as $x^2+xy=2xy^3$

Implicit Differentiation:-

"The process of finding dy/dx (derivative) from implicit functions is called implicit differentiation."

Exercise 2.4

Q.1 Find dy/dx in the functions:

(i) $y=\sqrt{\frac{1-x}{1+x}}$

Diff. w.r.t. x

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{1-x}{1+x}\right)^{1/2}$$

$$\frac{dy}{dx}=\frac{1}{2}\left(\frac{1-x}{1+x}\right)^{-1/2}\cdot\frac{d}{dx}\left(\frac{1-x}{1+x}\right)$$

$$\frac{dy}{dx}=\frac{1}{2}\left(\frac{1+x}{1-x}\right)^{1/2}\cdot\frac{(1+x)\frac{d}{dx}(1-x)-(1-x)\frac{d}{dx}(1+x)}{(1+x)^2}$$

$$\frac{dy}{dx}=\frac{1}{2}\frac{(1+x)^{1/2}}{(1-x)^{1/2}}\cdot\frac{(1+x)(0-1)-(0+1)(1-x)}{(1+x)^2}$$

$$\frac{dy}{dx}=\frac{1}{2}\frac{(x+1)^{1/2}}{\sqrt{1-x}}\cdot\frac{-1-x-1+x}{(1+x)^2}$$

$$\frac{dy}{dx}=\frac{-2}{2\sqrt{1-x}\,(1+x)^{2-1/2}}$$

$$\frac{dy}{dx}=\frac{-1}{\sqrt{1-x}\,(1+x)^{3/2}}\quad\text{Ans.}$$