2.5 Differentiation of Implicit Function

We have been concerned with the case in which y is expressed explicitly i.e. directly in terms of x. However, there are many cases in which y is not expressed directly in terms of x, but its functionality is implified by an algebraic relation connecting x and y.

2.5.1 Definition:

A function in which two or more variables are related to each other but are not independent to each other is called an implicit function.

An implicit function in x and y is expressed as f(x, y) = 0. For example:

(i) $x^2 - y^2 + xy + 3 = 0$ (ii) $x^3 + x^2y + y^3 = 7$

In such implicit relation we proceed as follows

Example 1:

If $x^3 + y^3 = 3axy$ find $\frac{dy}{dx}$

Solution:

Diff. both sides w.r.t.x.

$$3x^2 + 3y^2\frac{dy}{dx} = 3a\left(y.1 + x.\frac{dy}{dx}\right)$$

$$3(x^2 - ay) + 3(y^2 - ax)\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{(x^2 - ay)}{y^2 - ax}$$

Example 2:

Find $\frac{dy}{dx}$ from the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Solution: Diff. w.r.t.x.

$$2ax + 2h\left(x.\frac{dy}{dx} + y\right) + 2by.\frac{dy}{dx} + 2g + 2f.\frac{dy}{dx} = 0$$

$$(hx + by + f)\frac{dy}{dx} = -(ax + hy + g)$$

$$\frac{dy}{dx} = -\frac{ax + hy + g}{hx + by + f}$$

Where, $hx + by + f \neq 0$

2.6 Differentiation of Parametric Function:

Some time we deal with the functions in which dependent variable y is not given in terms of independent variable x rather both of them are functions of an other variable" t" called a parameter.

Its general form is $x = f(t)$ and $y = g(t)$

In order to differentiate parametric equations, we find $\frac{dx}{dt}$ and $\frac{dy}{dt}$ from the given equations, then using chain rule find $\frac{dy}{dx}$.

i.e. $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

Example 1:

Find $\frac{dy}{dx}$ when $x = at^2$, $y = 2at$

Solution:

Diff. both equations w.r.t.t.

As, $x = at^2 \implies \frac{dx}{dt} = 2at$

and $y = 2at \implies \frac{dy}{dt} = 2a$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = 2a \times \frac{1}{2at} = \frac{1}{t}$$

Example 2:

If $x = 2\theta + 1$, $y = (2\theta + 1)^2$ Find $\frac{dy}{dx}$

Solution:

Diff. both equation w.r.t. θ

$x = 2\theta + 1$, $y = (2\theta + 1)^2$

$\frac{dy}{d\theta} = 2$, $\frac{dy}{d\theta} = 2(2\theta + 1).2$

$= 4(2\theta + 1)$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$= 4(2\theta + 1) \times \frac{1}{2}$$

$$= 2(2\theta + 1)$$

Exercise 2.3

Q.1: Find $\frac{dy}{dx}$ of the following:

i. $xy + y^2 = 2$ ii. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

iii. $x^{2/3} + y^{2/3} = a^{2/3}$ iv. $x\sqrt{1+y} + y\sqrt{1+x} = a$

v. $x^3 - 4xy^2 - 2y^2 - 5y^3 = 0$ vi $\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}} = \frac{1}{\sqrt{a}}$

vii $x^2 - 3y^2 + 7 = 0$ viii $x^3 + y^3 + 4 = 0$

ix $x^2 - 2xy + 2y^2 = 5$ x $x^3 + 6xy + 5y^3 = 3$

Q.2: Find $\frac{dy}{dx}$ of the following:

i. $y^2 = 4ax$ ii. $x^3 + y^3 = xy$

iii. $x^5 + y^5 = 5a^2x^2y^2$ iv. $y^2 = \frac{x-1}{x+1}$

Q.3: Find $\frac{dy}{dx}$

i. $x = t + 2,\ \ y = 2t^2 + 2$

ii. $x = \theta^2 - \theta - 1,\ \ y = 2\theta^2 + \theta + 1$

iii. $x = a\theta^3,\ \ y = b\left(\theta - \frac{1}{\theta}\right)$ iv. $x = \frac{a(1 - t^2)}{1 + t^2},\ \ y = \frac{2bt}{1 + t^2}$

v. $x = \frac{3at}{1 + t^3},\ y = \frac{3at^2}{1 + t^3}$ vi. $x = a\left(\frac{t^2}{2} - t\right),\ y = b\left(\frac{t^3}{3} - \frac{t^2}{2}\right)$

vii. $x = \frac{1 - t^2}{1 + t^2},\ y = \frac{2t}{1 + t^2}$

Q.4: i. Show that if $x = a\theta^2$, $y = 2a\theta$ then $y\frac{dy}{dx} - 2a = 0$

ii. if $y = x^4 + 2x^2$, prove that $\frac{dy}{dx} = 4x\sqrt{y + 1}$

iii If $x = \frac{1 - t^2}{1 + t^2}$, $y = \frac{t^2}{1 + t^2}$ then prove that $y\frac{dy}{dx} + x = 0$

Answers 2.3

Q.1 (i) $\frac{-y}{x+2y}$ (ii) $-\frac{b^2x}{a^2y}$ (iii) $-\frac{y^{1/3}}{x^{1/3}}$

(iv) $\frac{-\sqrt{1+y}\left[y+2\sqrt{(1+x)(1+y)}\right]}{\sqrt{1+x}\left[x+2\sqrt{(1+x)(1+y)}\right]}$ (v) $\frac{3x^2-4y^2}{8xy+4y+15y^2}$

(vi) $-\left(\frac{y}{x}\right)^{3/2}$ (vii) $x/3y$ (viii) $-x^2/y^2$

(ix) $\frac{x-y}{x-2y}$ (x) $-\frac{x^2+2y}{2x+5y^2}$

Q.2 (i) $\frac{2a}{y}$ (ii) $\frac{3x^2-y}{x-3y^2}$ (iii) $\frac{x(2a^2y^2-x^3)}{y(y^3-2a^2x^2)}$

(iv) $\frac{1}{(x+1)^{3/2}\sqrt{x-1}}$

Q.3 (i) $4t$ (ii) $\frac{4\theta+1}{2\theta-1}$ (iii) $\frac{b}{3a}(\theta^{-2}+\theta^{-4})$

(iv) $\frac{b(t^2-1)}{2at}$ (v) $\frac{t(2-t^3)}{1-2t^3}$ (vi) $\frac{b}{a}t$

(vii) $\frac{t^2-1}{2t}$

2.7 Differentiation of Function w.r.t. Another Function

Sometime we differentiate given function w.r.t. another function instead of independent variable.

In this case we proceed as follows:

Example 1:

Differentiate x^3+8 w.r.t. x^2+4

Solution:

Let $y=x^3+8$ $u=x^2+4$