Exercise 2.10

Q.1 Let the integers are $x, 30-x$

Let P is the product of integers

$$P = x(30-x) = 30x - x^2$$

$$\frac{dP}{dx} = 30 - 2x$$

For extreme values $\frac{dP}{dx} = 0$

$$\Rightarrow 30 - 2x = 0 \Rightarrow x = 15$$

$$\frac{d^2P}{dx^2} = -2 < 0$$

So P is maximum at $x = 15$

So integers are $15, (30-15)$

integers are $15, 15$ Ans.

Q.2 Let x and $20-x$ are the parts

let Sum of Squares is S

$$S = x^2 + (20-x)^2$$

$$S = x^2 + 400 + x^2 - 40x$$

$$S = 2x^2 - 40x + 400$$

$$\frac{dS}{dx} = 4x - 40$$

For extreme values $\frac{dS}{dx} = 0$

$$\Rightarrow 4x - 40 = 0 \Rightarrow x = 10$$

$$\frac{d^2S}{dx^2} = 4 > 0$$

So "S" is minimum at $x = 10$

Thus parts are $10, (20-10)$

Parts are $10, 10$ Ans.

Q.3 Let the integers are $x, 12-x$

$$P = x^2 \cdot (12-x) = 12x^2 - x^3$$

$$\frac{dP}{dx} = 24x - 3x^2$$

For extreme values $\frac{dP}{dx} = 0$

$$\Rightarrow 24x - 3x^2 = 0 \Rightarrow x(24 - 3x) = 0$$

$$x = 0 \quad \wedge \quad x = 8$$

$$\frac{d^2P}{dx^2} = 24 - 6x$$

$$\left(\frac{d^2P}{dx^2}\right)_{x=0} = 24 > 0$$

P is minimum at $x = 0$

$$\left(\frac{d^2P}{dx^2}\right)_{x=8} = 24 - 6(8) = 24 - 48$$
$$= -24 < 0$$

P is maximum at $x = 8$

Thus integers are $8, (12-8)$

integers are $8, 4$ Ans.

Q.4 Let a, b, c be the Sides of $\triangle$

$2S = P = a+b+c = 16\text{ cm}$ Let $a = 6\text{cm}$

$\Rightarrow S = \frac{16}{2} = 8\text{cm}$

$$a+b+c = 16 \Rightarrow b+c = 16 - a$$

$$b+c = 16 - 6 = 10 \Rightarrow c = 10 - b$$

Thus A is area of triangle so

$$A = \sqrt{S(S-a)(S-b)(S-c)}$$

$$A = \sqrt{8(8-6)(8-b)(8-10+b)}$$

$$A = \sqrt{8 \times 2 \times (8-b) \times (b-2)}$$

$$A = \sqrt{16(10b - b^2 - 16)}$$

$$A = \sqrt{160b - 16b^2 - 256}$$

$A^2 = 160b - 16b^2 - 256$

Diff w.r.t. "b"

$2A\frac{dA}{db} = 160 - 32b$

$A\frac{dA}{db} = 80 - 16b \Rightarrow \frac{dA}{db} = \frac{80-16b}{A}$

For extreme values $\frac{dA}{db} = 0$

$\frac{80-16b}{A} = 0 \Rightarrow 80 - 16b = 0$

$b = \frac{80}{16} \Rightarrow b = 5\,cm$

$\frac{d}{db}\left[A\frac{dA}{db}\right] = \frac{d}{db}\left[80-16b\right]$

$A\cdot\frac{d^2A}{db^2} + \frac{dA}{db}\cdot\frac{dA}{db} = 0 - 16$

$A\frac{d^2A}{db^2} = -16 - \left(\frac{dA}{db}\right)^2$

$\frac{d^2A}{db^2} = \frac{-16 - \frac{80-16(b)}{A}}{A}$

$\left(\frac{d^2A}{db^2}\right)_{at\, b=5} = \frac{-16 - \frac{80-16(5)}{A}}{A^2}$

$\left(\frac{d^2A}{db^2}\right)_{at\, b=5} = \frac{-16 - 0/A}{A^2} = \frac{-16}{A^2}$

$\frac{d^2A}{db^2}$ **at b=5** $= \frac{-16}{A^2} < 0$

A is maximum at $b=5$

Thus sides are $5,\ 10-5$

Sides are $5, 5$

Q.5 Let x, y are the length and width

Perimeter $= 2(x+y)$

$120 = 2(x+y)$

$x + y = 60 \Rightarrow y = 60 - x$

Area = length × Width

$A = xy = x(60-x)$

$A = 60x - x^2$

$\frac{dA}{dx} = 60 - 2x$

For extreme values $\frac{dA}{dx} = 0$

$60 - 2x = 0 \Rightarrow 2x = 60$

$x = 30\ cm$

$\frac{d^2A}{dx^2} = -2 < 0$ at $x = 30$

A is maximum at $x = 30$

Thus dimensions are

length $(x) = 30\,cm$

Width $y = 60 - x = 60 - 30 = 30\,cm$

Q.6 Let x, y are the length and Width

$A = xy = 36\,cm^3$

$xy = 36 \Rightarrow y = \frac{36}{x}$

$P = 2[x+y]$ (P = Perimeter)

$P = 2\left(x + \frac{36}{x}\right) = 2\left[x + 36x^{-1}\right]$

$\frac{dP}{dx} = 2\left[1 - 36x^{-2}\right]$

For extreme values $\frac{dP}{dx} = 0$

$2\left(1 - \frac{36}{x^2}\right) = 0 \Rightarrow 1 - \frac{36}{x^2} = 0$

$1 = \frac{36}{x^2} \Rightarrow x^2 = 36 \Rightarrow x = \pm 6$

$\frac{d^2P}{dx^2} = 2(+72x^{-3}) = \frac{144}{x^3}$

$\frac{d^2P}{dx^2} < 0$ if $x = -6$

P is max. at $x = -6$

$$\frac{d^2P}{dx^2} > 0 \quad \text{at } x = 6$$

P is minimum at $x = 6$

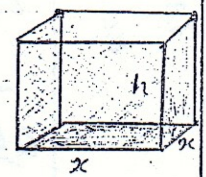
Thus Demensions are

length $x = 6$ cm

Width $y = \frac{36}{x} = \frac{36}{6} = 6$ cm

Q.7 Let x is length and Width while h is the height of box.

V = length×Width×height



$V = x \cdot x \cdot h = 4\ dm^3$

$x^2 h = 4\ dm^3$

$h = 4/x^2$

Let M be the material required.

M = Base Area + 4×(Side Area)

$M = x^2 + 4xh = x^2 + 4x(4/x^2)$

$M = x^2 + \frac{16}{x} = x^2 + 16x^{-1}$

$\frac{dM}{dx} = 2x - 16x^{-2}$

For extreme values $\frac{dM}{dx} = 0$

$2x - \frac{16}{x^2} = 0 \Rightarrow \frac{16}{x^2} = 2x$

$16 = 2x^3 \Rightarrow x^3 = 8 \Rightarrow x = 2$

$\frac{d^2M}{dx^2} = 2 + 32x^{-3} = 2 + \frac{32}{x^3} > 0$ at $x=2$

M is least (Minimum) at $x = 2$

Thus demensions are

length, Width $= (x) = 2$ dm

(h) height $= \frac{4}{(2)^2} = \frac{4}{4} = 1$ dm

Q.8 Let x, y are length and Width

Perimeter(P) = 80 m

$2[x+y] = P = 80$

$[x+y] = 40 \Rightarrow y = 40 - x$

$y = 40 - x$

Area (A) = length × Width

$A = xy = x(40-x)$

$A = 40x - x^2$

$\frac{dA}{dx} = 40 - 2x$

For extreme Values $\frac{dA}{dx} = 0$

$40 - 2x = 0 \Rightarrow x = 20$ m

$\frac{d^2A}{dx^2} = -2 < 0 \quad$ at $x = 20$

A is maximum at $x = 20$

Thus demensions are

length (x) = 20 m

Width (y) $= 40 - x = 40 - 20 = 20$ m

Q.9 Let h be the depth of tank and Q is the Quantity of water in Cubic unit

$V = Q = x^2 h$

$h = \frac{Q}{x^2}$

The interior Surface is S

S = Base Area + 4(Side Area)

$S = x^2 + 4xh = x^2 + 4x\left(\frac{Q}{x^2}\right)$

$S = x^2 + 4Qx^{-1}$

$\frac{dS}{dx} = 2x - 4Qx^{-2}$

For extreme values $\frac{dS}{dx} = 0$

$$2x - \frac{4Q}{x^2} = 0 \Rightarrow 2x = \frac{4Q}{x^2}$$

$$2x^3 = 4Q \Rightarrow x^3 = 2Q$$

$$x = (2Q)^{1/3}$$

$$\frac{d^2S}{dx^2} = 2 + 8Qx^{-3} = 2 + \frac{8Q}{x^3}$$

$$\frac{d^2S}{dx^2} = 2 + \frac{8Q}{[(2Q)^{1/3}]^3} = 2 + \frac{8Q}{2Q} = 2+4$$

$$\frac{d^2S}{dx^2} = 6 > 0 \quad \text{at } x = (2Q)^{1/3}$$

S is least (minimum) at $x = (2Q)^{1/3}$

Thus height (Depth) h is

$$h = \frac{Q}{x^2} = \frac{x^3}{2x^2} = \frac{x}{2}$$

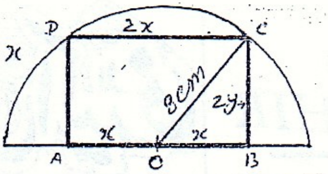
$$(\because x = (2Q)^{1/3} \Rightarrow \frac{x^3}{2} = Q)$$

Thus $h = \frac{x}{2}$ Ans.

Q.10 Let $2x$ and $2y$ are length and width of rectangle inscribed in semicircle.

$\overline{AB} = 2x$

But $AO = OB = \frac{AB}{2} = x$



From $\triangle OBC$

$$|OC|^2 = |OB|^2 + |BC|^2$$

$$(8)^2 = x^2 + 4y^2 \Rightarrow 4y^2 = 64 - x^2$$

$$y^2 = \frac{64-x^2}{4} \Rightarrow y = \frac{\sqrt{64-x^2}}{2}$$

Area of Rectangle = length × Width

$$A = (2x)(2y) = 4xy$$

$$A = 4x\frac{\sqrt{64-x^2}}{2} = 2x(64-x^2)^{1/2}$$

$$\frac{dA}{dx} = 2\cdot 1(64-x^2)^{1/2} + \frac{\cancel{2}x \cdot -2x}{\cancel{2}\sqrt{64-x^2}}$$

$$\frac{dA}{dx} = \frac{2(\sqrt{64-x^2})^2 - 2x^2}{\sqrt{64-x^2}}$$

$$\frac{dA}{dx} = \frac{2(64-x^2) - 2x^2}{\sqrt{64-x^2}} = \frac{128 - 2x^2 - 2x^2}{\sqrt{64-x^2}}$$

$$\frac{dA}{dx} = \frac{128-4x^2}{\sqrt{64-x^2}}$$

For extreme values $\frac{dA}{dx} = 0$

$$\frac{128-4x^2}{\sqrt{64-x^2}} = 0 \Rightarrow 128 - 4x^2 = 0$$

$$4x^2 = 128 \Rightarrow x^2 = 32 \Rightarrow x = 4\sqrt{2}$$

$$\frac{d^2A}{dx^2} = \frac{(\sqrt{64-x^2})(-8x) - (128-4x^2)\frac{-2x}{2\sqrt{64-x^2}}}{(\sqrt{64-x^2})^2}$$

$$\frac{d^2A}{dx^2} = \frac{\sqrt{64-(4\sqrt{2})^2}(-8(4\sqrt{2})) - 0}{(\sqrt{64-32})^2}$$

$$\frac{d^2A}{dx^2} = \frac{\sqrt{32}(-32\sqrt{2})}{32} < 0$$

A is maximum at $x = 4\sqrt{2}$

Thus dimensions are

length $(2x) = 2(4\sqrt{2}) = 8\sqrt{2}$ cm

$$4y^2 = 64 - (4\sqrt{2})^2 \Rightarrow 4y^2 = 64 - 32$$

$$4y^2 = 32 \Rightarrow y^2 = 8 \Rightarrow y = 2\sqrt{2}$$

Width $(2y) = 2(2\sqrt{2}) = 4\sqrt{2}$ cm

Q.11 Let $P(x, y)$ be the point closest to the $Q(3, -1)$ lying on $y = x^2 - 1$

Let S be the distance b/w P and Q

$$S = \sqrt{(x-3)^2 + (y+1)^2}$$

$$S = \sqrt{(x-3)^2 + (x^2-1+1)^2} = \sqrt{x^2 + 9 - 6x + x^4}$$

$$\frac{dS}{dx} = \frac{4x^3 + 2x - 6}{2\sqrt{x^4 + x^2 - 6x + 9}}$$

For extreme values $\frac{dS}{dx}=0$

$$\frac{4x^3+2x-6}{2\sqrt{x^4+x^2-6x+9}}=0$$

$$\Rightarrow 2(2x^3+x-3)=0 \qquad 2\neq 0$$

$$2x^3+x-3=0$$

Let $x=1$, $x-1=0$

	2	0	1	−3
1		2	2	3
	2	2	3	0

Thus

$$2x^3+x-3=(x-1)(2x^2+2x+3)$$

$x-1=0 \wedge 2x^2+2x+3=0$

$x=1$ — Gives imaginary roots.

$$\frac{d^2S}{dx^2}>0 \quad at\ x=1$$

Thus S is minimum at $x=1$

So $y=x^2-1=1^2-1=0$

Thus Point $P(x,y)=P(1,0)$

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Q.12 Let $P(x,y)$ be the point closest to the point $Q(18,1)$ lying on $y=x^2+1$

Let S is the distance b/w P and Q.

$$S=\sqrt{(x-18)^2+(y-1)^2}$$

$$S=\sqrt{x^2+324-36x+(x^2+1-1)^2}$$

$$S=\sqrt{x^2+324-36x+x^4}$$

$$\frac{dS}{dx}=\frac{4x^3+2x-36}{2\sqrt{x^4+x^2-36x+324}}$$

For extreme values $\frac{dS}{dx}=0$

$$\frac{2(2x^3+x-18)}{2\sqrt{x^4+x^2-36x+324}}=0$$

$$2x^3+x-18=0$$

Let $x=2$

	2	0	1	−18
2		4	8	18
	2	4	9	0

$x=2 \wedge 2x^2+4x+9=0$

$x=2$ — Gives imaginary value

$$\frac{d^2S}{dx^2}>0 \quad at\ x=2$$

S is minimum at $x=2$

$$y=(2)^2+(1)=4+1=5$$

Thus $P(x,y)=P(2,5)$ is Closest to $Q(18,1)$
