Chapter 2

Derivatives

2.1 Introduction:

Derivatives are used widely in science, economics, medicine and computer science to calculate velocity and acceleration, to explain the behavior of machinery, to estimate the drop in water levels as water is pumped out of tank and to predict the consequences of making cross in measurements.

Finding derivatives by evaluating limits can be lengthy and difficult. In this chapter we develop techniques to make calculating derivatives easier.

Increments :

If a variable x changes from one fixed value x_1 to another x_2, the difference $x_2 - x_1$ is called an **increment** of x. In general an incement of x may be positive or negative , and it is denoted by the symbol δx, and read as "delta x". Similarly, δy denotes as incement of y

2.2 Definition of a Derivative:

Let $y = f(x)$ be a function of x , then

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$, if it exists.

is called the **differential coefficient or derivative** of y with respect to x and it is denoted by $\frac{dy}{dx}$ or y' or $f'(x)$ or $y^{(1)} = D_y$, where $D = \frac{d}{dx}$

The steps for obtaining a derivative are as follows:

Consider the function $y = f(x)$

1. Give increments to x and y to obtain $y + \delta y = f(x + \delta x)$
2. Subtract to obtain $\delta y = f(x + \delta x) - f(x)$
3. Divide by δx to obtain $\frac{\delta y}{\delta x} = \frac{f(x + \delta y) - f(x)}{\delta x}$
4. Take the Limit to obtain $\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$

If the limit exists it is called the derivative or the differential coefficient of the function f(x) w.r.t. x, it is denoted by $\frac{dy}{dx}$ or $f'(x)$.

The process to find the derivative from the above four steps is called differentiation by ab-initio method or by definition or from first principles.

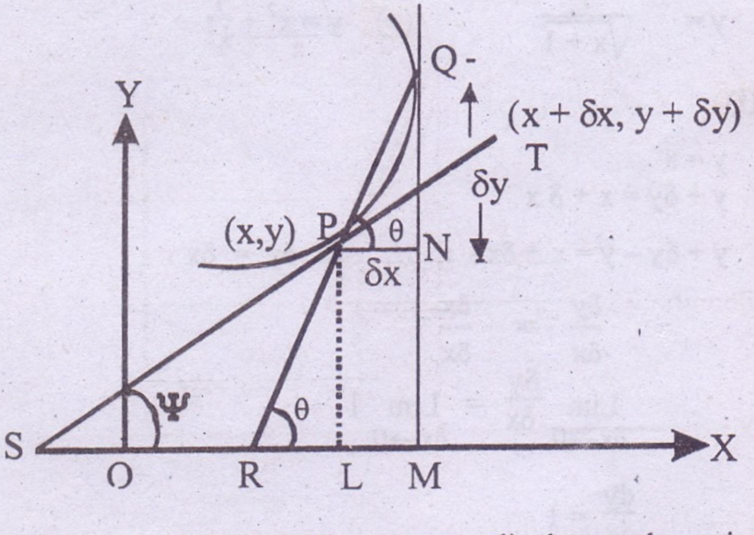
Secant Line:

Any line through two points on a curve is called a secant line.

2.3 Geometrical Interpretation of a derivative – Definition of a Derivative

If y = f(x), then the value of dy/dx at x = a is the slope of the Tangent of the curve represented by y = f(x).

Let P (x, y) and Q ($x + \delta x$, $y + \delta y$) be any two points on the curve y = f(x). Join QP and produce it to meet the x-axis at R.



PL, QM and PN have been drawn perpendiculars as shown in the figure. Clearly, PN = δx, NQ = δy.

If $\angle QPN = \theta$ then $\tan\theta = \frac{NQ}{PN} = \frac{\delta y}{\delta x}$. As $\delta x \to 0$, $Q \to P$ and the line QPR becomes tangent at P. Suppose the angle that the tangent at P makes with the x-axis at S is denoted by Ψ, then

$$\tan\Psi = \underset{Q\to P}{\text{Lt}} \tan\theta = \underset{\delta x\to 0}{\text{Lt}} \frac{\delta y}{\delta x} = \frac{dy}{dx}$$

i.e., slope (or gradient) of the tangent at $P(x,y) = \frac{dy}{dx}$, the value of the differential co-efficient at P(x, y). This way of stating the slope of a curve is called Leibnitz notation.

$$\text{The slope of the curve at } P = \lim_{\delta x \to 0} \frac{f(x+\delta x)-f(x)}{\delta x} = f'(x)$$

This way of stating the slope of a curve is called Functional notation.

Examples1:

Differentiate by first principle (ab-initio) w.r.t. x

(i) $y = x$ (ii) $y = \frac{1}{x^3}$ (iii). $y = \sqrt{x}$

(iv). $y = \frac{1}{\sqrt{x+1}}$ (v). $y = x^2 + \frac{1}{x^2}$

Solution: (i)

$$y = x$$
$$y + \delta y = x + \delta x$$
$$y + \delta y - y = x + \delta x - x \quad \Rightarrow \delta y = \delta x$$
$$\frac{\delta y}{\delta x} = \frac{\delta x}{\delta x}$$
$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} 1$$
$$\frac{dy}{dx} = 1$$

Solution: (ii) $y = \frac{1}{x^3}$

$$y = \frac{1}{x^3}$$
$$y + \delta y = \frac{1}{(x+\delta x)^3}$$
$$y + \delta y - y = \frac{1}{(x+\delta x)^3} - \frac{1}{x^3}$$

$$\delta y = \frac{x^3 - [x^3 + 3x^2(\delta x) + 3x(\delta x)^2 + (\delta x)^3]}{x^3(x+\delta x)^3}$$

$$= \frac{-3x^2\delta x - 3x(\delta x)^2 - (\delta x)^3}{x^3(x+\delta x)^3}$$

$$\frac{\delta y}{\delta x} = \frac{-3x^2 - 3x(\delta x) - (\delta x)^2}{x^3(x+\delta x)^3}$$

$$\lim_{x\to\infty}\frac{\delta y}{\delta x} = \lim_{x\to 0}\frac{-3x^2 - 3x(\delta x) - (\delta x)^2}{x^3(x+\delta x)^3}$$

$$\frac{dy}{dx} = \lim_{x\to 0}\frac{-3x^2}{x^6} = -\frac{3}{x^4}$$

Solution (iii) $y = \sqrt{x}$

$$y + \delta y = \sqrt{x+\delta x}$$

$$y + \delta y - y = \sqrt{x+\delta x} - \sqrt{x}$$

$$\delta y = \sqrt{x+\delta x} - \sqrt{x}$$

$$\frac{\delta y}{\delta x} = \frac{\sqrt{X+\delta X} - \sqrt{X}}{\delta X}$$

$$\frac{\delta y}{\delta x} = \frac{\sqrt{X+\delta X} - \sqrt{X}}{\delta X} \times \frac{\sqrt{X+\delta X} + \sqrt{X}}{\sqrt{X+\delta X} + \sqrt{X}} \quad \text{(By rationalization)}$$

$$= \frac{X+\delta X - X}{\delta X(\sqrt{X+\delta X} + \sqrt{X})}$$

$$\lim_{x\to 0}\frac{\delta y}{\delta x} = \lim_{x\to 0}\frac{1}{\sqrt{x+\delta x}+\sqrt{x}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

Solution : (iv). $y = \frac{1}{\sqrt{x+1}} \Rightarrow y + \delta y = \frac{1}{\sqrt{x+\delta x+1}}$

$$y + \delta y - y = \frac{1}{\sqrt{x+\delta x+1}} - \frac{1}{\sqrt{x+1}}$$

$$\delta y = \frac{\sqrt{x+1} - \sqrt{x+\delta x+1}}{\sqrt{x+\delta x+1}\ \sqrt{x-1}} \times \frac{\sqrt{x+1} + \sqrt{x+\delta x+1}}{\sqrt{x+1} + \sqrt{x+\delta s+1}}$$

$$\delta y = \frac{x+1-x-\delta x-1}{\sqrt{x+\delta x+1}\,.\,\sqrt{x+1}\,(\sqrt{x+1})+\sqrt{x+\delta x+1})}$$

$$\lim_{x\to 0}\frac{\delta y}{\delta x} = \lim_{x\to 0}\frac{-1}{\sqrt{x+\delta x+1}\sqrt{x+1}\,[\sqrt{x+1}+\sqrt{x+\delta x+1}]}$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{x+1}\sqrt{x+1}\,[\sqrt{x+1}+\sqrt{x+1}]}$$

$$\frac{dy}{dx} = \frac{-1}{(x+1)^2\,(2\sqrt{x+1})\,)} = \frac{-1}{2(x+1)^{3/2}}$$

Solution: (v). $y = x^2 + \frac{1}{x^2}$

$$y + \delta y = (x+\delta x)^2 + \frac{1}{(x+\delta x)^2}$$

$$y + \delta y - y = (x+\delta x)^2 + \frac{1}{(x+\delta x)^2} - x^2 - \frac{1}{x^2}$$

$$\delta y = (x+\delta x)^2 + \frac{1}{(x+\delta x)^2} - x^2 - \frac{1}{x^2}$$

$$(x+\delta x)^2 - x^2 + \frac{1}{(x+\delta x)^2} - \frac{1}{x^2}$$

$$\delta y = x^2 + (\delta x)^2 + 2x\,(\delta x) - x^2 + \frac{x^2-(x+\delta x)^2}{(x+\delta x)^2\,x^2}$$

$$\delta y = x^2 + (\delta x)^2 + 2x\,(\delta x) + \frac{x^2 - x^2 - (\delta x)^2 - 2x(\delta x)}{(x+\delta x)^2\,x^2}$$

$$\delta y = (\delta x)^2 + 2x(\delta x) - \frac{\delta x\,(\delta x + \delta x)}{(x+\delta x)^2\,x^2}$$

$$\frac{\delta y}{\delta x} = \frac{\delta x[(\delta x + 2x)]}{\delta x} - \frac{\delta x\,(\delta x+2x)}{\delta x\,(x+\delta x)^2\,x^2}$$

$$\lim_{x\to 0}\frac{\delta y}{\delta x} = \lim_{x\to 0}(\delta x + 2x) - \lim_{x\to 0}\frac{\delta x + 2x}{(x+\delta x)^2\,x^2}$$

$$\frac{dy}{dx} = 2x - \frac{2x}{x^2x^2} = 2x - \frac{2x}{x^4}$$

$$\frac{dy}{dx} = 2x - \frac{2}{x^3}$$

Exercise 2.1

Q.1: Differentiate w.r.t x ab-initio.

i. x^3 ii. $\frac{1}{x^2}$ iii. $\frac{1}{\sqrt{x}}$

iv. $x^{\frac{2}{3}}$ v. $x^{\frac{1}{4}}$ vi. $x^{-\frac{4}{5}}$

Q.2: Find the derivative from first principle.

i. $x^3 - 3x + 4$ ii. $\sqrt{x+9}$ iii $(x+4)^{\frac{2}{3}}$

Answers 2.1

Q.1: i. $3x^2$ ii. $\frac{-2}{x^3}$ iii. $-\frac{1}{2x^{3/2}}$

iv. $\frac{2}{3x^{1/3}}$ v. $\frac{1}{4x^{3/4}}$ vi. $-\frac{4}{5}x^{-9/5}$

Q.2 i. $3x^2 - 3$ ii. $\frac{1}{2\sqrt{x+9}}$ iii $\frac{2}{3}(x+4)^{-\frac{1}{3}}$

2.4 Fundamental Rules for Differentiation

Rule 1:

Derivative of x^n (power rule)

Let $y = x^n$

$$y + \delta y = (x + \delta x)^n$$

$$\delta y = (x + \delta x)^n - x^n \quad = x^n \left(1 - \frac{\delta x}{x} \right)^n - x^n$$

$$= x^n\left[(1 + \frac{\delta x}{x})^n - 1\right]$$

$$= x^n\left[1 + n\frac{\delta x}{x} + \frac{n(n-1)}{2!}\left(\frac{\delta x}{x}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{\delta x}{x}\right)^3 + \ldots\ldots - 1\right]$$