DIFFERENTIATION

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Calculus:-

"The branch of mathematics which deals with the change of a variable with respect to another variable is called calculus."

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Calculus is divided into two branches:

(i) Differential Calculus. (ii) Integral Calculus.

(i) Differential Calculus:-

"This branch of calculus is used to study the change of a function and used to find the change of function is called Differential Calculus."

(ii) Integral Calculus:-

"The branch of calculus which used to find out the functions from their change called Integral Calculus."

Derivative of a Function:-

Let f is a real valued continuous in the interval $[x, x+\Delta x] \in Dom(f)$ then $f'(x)$ is called derivative and is defined as

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} \qquad (\because f'(x) = \text{"f" prime of } x)$$

If $y = f(x)$ is a function then derivative of the function is denoted as $\frac{dy}{dx}$ or $y' = f'(x)$

* The derivative of a function exists if function is continuous at given interval belonging to the $Dom(f)$.
* "The process of finding derivative is called differentiation."

Derivative of a function by defination:-

If $y = f(x)$ is a Continuous then the derivative of function by defination is consists of the following steps:

Step I: $y = f(x)$ —— (1) $\quad y + \delta y = f(x + \delta x)$ —— (2)

Step II: Subtracting Equation (1) from Equation (2), we have

$$\delta y = f(x + \delta x) - f(x)$$

Step III: Dividing both sides by δx where $\lim\limits_{\delta x \to 0}$

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$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} \text{ ——— (A)}$$

Equation (A) is called derivative of $f(x)$ and the process is called by defination derivative, ~~by~~ first principle derivative or by ab-initio method to derivative.

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Exercise 2·1

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Find the derivatives of the following functions by defination:

(i) $2x^2 + 1$

Let $y = 2x^2 + 1$ —— (1)

Now $y + \delta y = 2(x + \delta x)^2 + 1$ —— (2)

Subtracting Eq (1) from Eq (2)

$$\delta y = 2(x+\delta x)^2 + 1 - 2x^2 - 1$$

$$\delta y = 2[x^2 + \delta x^2 + 2x\delta x] - 2x^2$$

$$\delta y = \cancel{2x^2} + 2\delta x^2 + 4x\delta x - \cancel{2x^2}$$

$$\delta y = \delta x(4x + 2\delta x)$$

Dividing both sides by δx where $\lim\limits_{\delta x \to 0}$

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{\delta x[4x + 2\delta x]}{\delta x}$$

$$\frac{dy}{dx} = \lim_{\delta x \to 0} (4x + 2\delta x)$$

By Applying Limit, we have

$$\frac{dy}{dx} = 4x + 2(0) = 4x + 0$$

$$\boxed{\frac{dy}{dx} = 4x}$$

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* $\lim\limits_{\delta x \to 0} \dfrac{\delta y}{\delta x} = \dfrac{dy}{dx}$
* δx and δy are called increment of x and y respectively.
* d[illegible] are called differentials [illegible] respectively.

(ii) $2-\sqrt{x}$

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Let $y = 2-\sqrt{x}$ ——(1)

Now $y+\delta y = 2-\sqrt{(\delta x+x)}$ ——(2)

Subtracting Eq (1) from Eq (2)

$$\delta y = 2-\sqrt{\delta x+x} - 2+\sqrt{x}$$
$$\delta y = \sqrt{x}-\sqrt{x+\delta x}$$
$$\delta y = \sqrt{x}-\sqrt{x}\left(\sqrt{1+\frac{\delta x}{x}}\right)$$
$$\delta y = \sqrt{x}-\sqrt{x}\left(1+\frac{\delta x}{x}\right)^{1/2}$$

Using Binomial theorem

$$\delta y = \sqrt{x}-\sqrt{x}\left[1+\frac{\delta x}{2x}+\frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}\left(\frac{\delta x}{x}\right)^2+\cdots\right]$$
$$\delta y = \cancel{\sqrt{x}}-\cancel{\sqrt{x}}-\frac{\delta x}{2\sqrt{x}}+\frac{1}{8}\frac{(\delta x)^2}{x^{3/2}}+\cdots$$
$$\delta y = \delta x\left(-\frac{1}{2\sqrt{x}}+\frac{\delta x}{8x^{3/2}}+\cdots\right)$$

Dividing both sides by δx where $\lim\limits_{\delta x\to 0}$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{\delta x\left[\frac{-1}{2\sqrt{x}}+\frac{\delta x}{8x^{3/2}}+\cdots\right]}{\delta x}$$
$$\frac{dy}{dx} = \lim_{\delta x\to 0}\left(\frac{-1}{2\sqrt{x}}+\frac{\delta x}{8x^{3/2}}+\cdots\right)$$

By Applying Limit, we have

$$\frac{dy}{dx} = \frac{-1}{2\sqrt{x}}+\frac{(0)}{8x^{3/2}}+\cdots$$
$$= \frac{-1}{2\sqrt{x}}+0+0+\cdots$$
$$\boxed{dy/dx = -1/2\sqrt{x}}$$

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(iii) $\frac{1}{\sqrt{x}}$

Let $y = \frac{1}{\sqrt{x}}$ ——(1)

$y+\delta y = \frac{1}{\sqrt{x+\delta x}}$ ——(2)

Subtracting Eq (1) from Eq (1)

$$\delta y = \frac{1}{\sqrt{x+\delta x}}-\frac{1}{\sqrt{x}} = \frac{\sqrt{x}-\sqrt{x+\delta x}}{\sqrt{x+\delta x}\sqrt{x}}$$

$$\delta y = \frac{\sqrt{x}-\sqrt{x}\left(1+\frac{\delta x}{x}\right)^{1/2}}{\sqrt{x}\sqrt{x+\delta x}}$$
$$\delta y = \frac{\cancel{\sqrt{x}}\left[1-\left(1+\frac{\delta x}{x}\right)^{1/2}\right]}{\cancel{\sqrt{x}}\sqrt{x+\delta x}}$$
$$\delta y = \frac{1-\left(1+\frac{\delta x}{x}\right)^{1/2}}{\sqrt{x+\delta x}}$$

Using binomial theorem for $\left(1+\frac{\delta x}{x}\right)^{1/2}$

$$\delta y = \frac{1-\left(1+\frac{\delta x}{2x}+\frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}\left(\frac{\delta x}{x}\right)^2+\cdots\right]}{\sqrt{x+\delta x}}$$
$$\delta y = \frac{\cancel{1}-\cancel{1}-\frac{\delta x}{2x}+\frac{\delta x^2}{8x^2}+\cdots}{\sqrt{x+\delta x}}$$
$$\delta y = \frac{\delta x\left(\frac{-1}{2x}+\frac{\delta x}{8x^2}+\cdots\right)}{\sqrt{x+\delta x}}$$

Dividing both sides by δx where $\lim\limits_{\delta x\to 0}$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{\delta x\left[\frac{-1}{2x}+\frac{\delta x}{8x^2}+\cdots\right]}{\delta x\sqrt{x+\delta x}}$$
$$\frac{dy}{dx} = \lim_{\delta x\to 0}\frac{\left(-\frac{1}{2x}+\frac{\delta x}{8x^2}+\cdots\right)}{\sqrt{x+\delta x}}$$

By Applying Limit

$$\frac{dy}{dx} = \frac{\frac{-1}{2x}+0+0+\cdots}{\sqrt{x+0}} = \frac{\frac{-1}{2x}}{\sqrt{x}}$$
$$\frac{dy}{dx} = \frac{-1}{2x\sqrt{x}} = \frac{-1}{2x^{3/2}}$$
$$\boxed{\frac{dy}{dx} = \frac{-1}{2x^{3/2}}}$$

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(iv) $\dfrac{1}{x^3}$

Let $y = \dfrac{1}{x^3}$ —— (1)

$$y+\delta y = \frac{1}{(x+\delta x)^3} \quad \text{—— (2)}$$

Subtracting Eq (1) from Eq (2)

$$\delta y = \frac{1}{(x+\delta x)^3} - \frac{1}{x^3}$$

$$\delta y = \frac{x^3-(x+\delta x)^3}{x^3(x+\delta x)^3}$$

$$\delta y = \frac{x^3-(x^3+\delta x^3+3x^2\delta x+3x\delta x^2)}{x^3(x+\delta x)^3}$$

$$\delta y = \frac{\cancel{x^3}-\cancel{x^3}-\delta x^3-3x^2\delta x-3x\delta x^2}{x^3(x+\delta x)^3}$$

$$\delta y = \frac{\delta x(-\delta x^2-3x^2-3x\delta x)}{x^3(x+\delta x)^3}$$

Dividing both sides by δx where $\lim\limits_{\delta x\to 0}$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{\cancel{\delta x}(-3x^2-3x\delta x-\delta x^2)}{\cancel{\delta x}\;x^3(x+\delta x)^3}$$

$$\frac{dy}{dx} = \lim_{\delta x\to 0}\frac{-3x^2-3x\delta x-\delta x^2}{x^3(x+\delta x)^3}$$

By Applying Limit, we have

$$\frac{dy}{dx} = \frac{-3x^2-3x(0)-(0)^2}{x^3(x+0)^3} = \frac{-3x^2}{x^6}$$

$$\boxed{\frac{dy}{dx} = \frac{-3}{x^4}}$$

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(v) $\dfrac{1}{x-a}$

Let $y = \dfrac{1}{x-a}$ —— (1)

$$y+\delta y = \frac{1}{(x+\delta x)-a} \quad \text{—— (2)}$$

Subtracting Eq (1) from (2)

$$\delta y = \frac{1}{(x+\delta x-a)} - \frac{1}{(x-a)}$$

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$$\delta y = \frac{\cancel{x}-\cancel{a}-x-\delta x+\cancel{a}}{(x-a)(x+\delta x-a)}$$

$$\delta y = \frac{-\delta x}{(x-a)(x+\delta x-a)}$$

Dividing by δx where $\lim\limits_{\delta x\to 0}$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{-\cancel{\delta x}}{\cancel{\delta x}(x-a)(x+\delta x-a)}$$

$$\frac{dy}{dx} = \lim_{\delta x\to 0}\frac{-1}{(x-a)(x+\delta x-a)}$$

By Applying Limit, we have

$$\frac{dy}{dx} = \frac{-1}{(x-a)(x+0-a)} = \frac{-1}{(x-a)^2}$$

$$\boxed{\frac{dy}{dx} = \frac{-1}{(x-a)^2}}$$

(vi) $x(x-3) = x^2-3x$

Let $y = x^2-3x$ —— (1)

$$y+\delta y = (x+\delta x)^2-3(x+\delta x) \quad \text{—— (2)}$$

Subtracting Eq (1) from Eq (2)

$$\delta y = \cancel{x^2}+\delta x^2+2x\delta x-\cancel{3x}-3\delta x-\cancel{x^2}+\cancel{3x}$$

$$\delta y = 2x\delta x+\delta x^2-3\delta x$$

$$\delta y = \delta x(2x+\delta x-3)$$

Dividing by δx where $\lim\limits_{\delta x\to 0}$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{\cancel{\delta x}(2x+\delta x-3)}{\cancel{\delta x}}$$

$$\frac{dy}{dx} = \lim_{\delta x\to 0} 2x+\delta x-3$$

By Applying Limit

$$\frac{dy}{dx} = 2x+0-3 = 2x-3$$

$$\boxed{\frac{dy}{dx} = 2x-3}$$

Next Part is similar to (iv)

(viii) x^2+1/x^2

Let $y = x^2 + 1/x^2$ ——(1)

$y+\delta y = (x+\delta x)^2 + \frac{1}{(x+\delta x)^2}$ ——(2)

Subtracting Eq (1) from Eq (2)

$\delta y = (\cancel{x^2}+\delta x^2+2x\delta x) + \frac{1}{(x^2+\delta x^2+2x\delta x)} - \cancel{x^2} - 1/x^2$

$\delta y = 2x\delta x + \delta x^2 + \frac{1}{x^2+\delta x^2+2x\delta x} - \frac{1}{x^2}$

$\delta y = 2x\delta x+\delta x^2 + \frac{\cancel{x^2} - \cancel{x^2} - \delta x^2 - 2x\delta x}{x^2(x^2+\delta x^2+2x\delta x)}$

$\delta y = \delta x\left(2x+\delta x - \frac{\delta x+2x}{x^2(x^2+\delta x^2+2x\delta x)}\right)$

Dividing by δx where $\lim\limits_{\delta x\to 0}$

$\lim\limits_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim\limits_{\delta x\to 0}\frac{\cancel{\delta x}\left(2x+\delta x - \frac{\delta x+2x}{x^2(x^2+\delta x^2+2x\delta x)}\right)}{\cancel{\delta x}}$

$\frac{dy}{dx} = \lim\limits_{\delta x\to 0}\left(2x+\delta x - \frac{2x+\delta x}{x^2(x^2+\delta x^2+2x\delta x)}\right)$

By Applying Limit

$\frac{dy}{dx} = 2x+0 - \frac{2x+0}{x^2(x^2+0+0)} = 2x - \frac{2x}{x^4}$

$$\boxed{\frac{dy}{dx} = 2x - \frac{2}{x^3}}$$

(ix) $(x+3)^{1/3}$

Let $y = (x+3)^{1/3}$ ——(1)

$y+\delta y = (x+\delta x+3)^{1/3}$ ——(2)

Subtracting Eq (1) from Eq (2)

$\delta y = [(x+3)+\delta x]^{1/3} - (x+3)^{1/3}$

$\delta y = (x+3)^{1/3}\left[1+\frac{\delta x}{(x+3)}\right]^{1/3} - (x+3)^{1/3}$

using binomial theorem for $\left(1+\frac{\delta x}{x+3}\right)$

$\delta y = (x+3)^{1/3}\left[1+\frac{\delta x}{3(x+3)} + \frac{\frac{1}{3}(\frac{1}{3}-1)}{2!}\frac{\delta x^2}{(x+3)^2}+\cdots\right] - (x+3)^{1/3}$

$\delta y = \cancel{(x+3)^{1/3}} + \frac{\delta x}{3(x+3)^{2/3}} - \frac{2\delta x^2}{9(x+3)^{5/3}} + \cdots - \cancel{(x+3)^{1/3}}$

$\delta y = \delta x\left(\frac{1}{3(x+3)^{2/3}} - \frac{2\delta x}{9(x+3)^{5/3}}+\cdots\right)$

Dividing by δx where $\lim\limits_{\delta x\to 0}$

$\lim\limits_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim\limits_{\delta x\to 0}\frac{\cancel{\delta x}\left[\frac{1}{3(x+3)^{2/3}} - \frac{2\delta x}{9(x+3)^{5/3}}+\cdots\right]}{\cancel{\delta x}}$

$\frac{dy}{dx} = \lim\limits_{\delta x\to 0}\left(\frac{1}{3(x+3)^{2/3}} - \frac{2\delta x}{9(x+3)^{5/3}}+\cdots\right)$

By Applying Limit

$\frac{dy}{dx} = \frac{1}{3[x+3]^{2/3}} + 0 + 0 \cdots$

$$\boxed{\frac{dy}{dx} = \frac{1}{3(x+3)^{2/3}}}$$

(x) $x^{3/2}$

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Let $y = x^{3/2}$ ——(1)

$y+\delta y = (x+\delta x)^{3/2}$ ——(2)

Subtracting Eq (1) from Eq (2)

$\delta y = (x+\delta x)^{3/2} - x^{3/2}$

$\delta y = x^{3/2}\left(1+\frac{\delta x}{x}\right)^{3/2} - x^{3/2}$

Using binomial theorem

$\delta y = x^{3/2}\left[1+\frac{3\delta x}{2x} + \frac{\frac{3}{2}(\frac{3}{2}-1)}{2!}\frac{\delta x^2}{x^2}+\cdots\right] - x^{3/2}$

$\delta y = \cancel{x^{3/2}} + \frac{3}{2}x^{1/2}\delta x + \frac{3}{8}\frac{\delta x^2}{x^{1/2}}+\cdots - \cancel{x^{3/2}}$

$\delta y = \delta x\left[\frac{3}{2}x^{1/2} + \frac{3}{8}\frac{\delta x}{x^{1/2}}+\cdots\right]$

Dividing by δx where $\lim\limits_{\delta x\to 0}$

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$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\frac{\cancel{\delta x}\left(\frac{3}{2}x^{1/2}+\frac{3}{8}\frac{\delta x}{x^{1/2}}+\cdots\right)}{\cancel{\delta x}}$$

$$\frac{dy}{dx}=\lim_{\delta x\to 0}\left(\frac{3}{2}x^{1/2}+\frac{3}{8}\frac{\delta x}{x^{1/2}}+\cdots\right)$$

By Applying Limit

$$\frac{dy}{dx}=\frac{3}{2}x^{1/2}+0+\cdots$$

$$\boxed{\frac{dy}{dx}=\frac{3}{2}\sqrt{x}}$$

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Similarly $x^{5/2}$ is to Solve.

(xii) x^m

Let $y=x^m$ ——(1)

$y+\delta y=(x+\delta x)^m$ ——(2)

Subtracting Eq (1) from Eq (2)

$$\delta y=(x+\delta x)^m-x^m$$

$$\delta y=x^m\left(1+\frac{\delta x}{x}\right)^m-x^m$$

Using binomial theorem for $\left(1+\frac{\delta x}{x}\right)^m$

$$\delta y=x^m\left[1+\frac{m\delta x}{x}+\frac{m(m-1)}{2!}\frac{\delta x^2}{x^2}+\cdots\right]-x^m$$

$$\delta y=\cancel{x^m}+mx^{m-1}\delta x+\frac{m(m-1)}{2!}x^{m-2}\delta x^2+\cdots-\cancel{x^m}$$

$$\delta y=\delta x\left(mx^{m-1}+\frac{m(m-1)}{2!}x^{m-2}\delta x+\cdots\right)$$

Dividing by δx where $\lim_{\delta x\to 0}$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\frac{\cancel{\delta x}\left(mx^{m-1}+\frac{m(m-1)}{2!}x^{m-2}\delta x+\cdots\right)}{\cancel{\delta x}}$$

$$\frac{dy}{dx}=\lim_{\delta x\to 0}\left(mx^{m-1}+\frac{m(m-1)}{2!}\delta x\, x^{m-2}+\cdots\right)$$

By Applying Limit

$$\frac{dy}{dx}=mx^{m-1}+0+\cdots$$

$$\boxed{\frac{dy}{dx}=mx^{m-1}}$$

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(xiii) $\frac{1}{x^m}=x^{-m}$

Let $y=x^{-m}$ ——(1)

$y+\delta y=(x+\delta x)^{-m}$ ——(2)

Subtracting Eq (1) from Eq (2)

$$\delta y=(x+\delta x)^{-m}-x^{-m}$$

$$\delta y=x^{-m}\left(1+\frac{\delta x}{x}\right)^{-m}-x^{-m}$$

Using binomial theorem for $\left(1+\frac{\delta x}{x}\right)^{-m}$

$$\delta y=x^{-m}\left[1-\frac{m\delta x}{x}-\frac{m(-m-1)}{2!}\frac{\delta x^2}{x^2}+\cdots\right]-x^{-m}$$

$$\delta y=\cancel{x^{-m}}-mx^{-m-1}\delta x+\frac{m(m+1)}{2!}x^{-m-2}\delta x^2+\cdots-\cancel{x^{-m}}$$

$$\delta y=\delta x\left(-mx^{-m-1}+\frac{m(m+1)}{2!}x^{-m-2}\delta x+\cdots\right)$$

Dividing by δx where $\lim_{\delta x\to 0}$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\frac{\cancel{\delta x}\left(-x^{-m-1}m+\frac{m(m+1)}{2!}x^{-m-2}\delta x+\cdots\right)}{\cancel{\delta x}}$$

$$\frac{dy}{dx}=\lim_{\delta x\to 0}\left(-\frac{m}{x^{m+1}}+\frac{m(m+1)}{2!}\frac{\delta x}{x^{m+2}}+\cdots\right)$$

By Applying Limit

$$\frac{dy}{dx}=\frac{-m}{x^{m+1}}+0+\cdots$$

$$\boxed{\frac{dy}{dx}=\frac{-m}{x^{m+1}}}$$

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Similarly x^{40}, x^{-100} to Solve.

Q.2 (i) $\sqrt{x+2}=(x+2)^{1/2}$

This part is similar to $(x+4)^{1/3}$ do this question yourself.

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