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M.Sc Math
0345-6510779

1st Year
(Mathematics)

Chapter: No. 13

INVESE TRIGONOMETRIC FUNCTIONS.

Inverse Trigonometric Functions:

If trigonometric metric functions are one-one in their domain then inverse trigonometric functions are defined as:

Inverse Trigonometric Functions	Domains	Ranges
$y = \sin^{-1}x \Leftrightarrow x = \sin y$	$-1 \le x \le 1$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$
$y = \cos^{-1}x \Leftrightarrow x = \cos y$	$-1 \le x \le 1$	$[0, \pi]$
$y = \tan^{-1}x \Leftrightarrow x = \tan y$	$(-\infty, \infty)$ or $\mathbb{R}$	$(-\frac{\pi}{2}, \frac{\pi}{2})$
$y = \cot^{-1}x \Leftrightarrow x = \cot y$	$(-\infty, \infty)$ or $\mathbb{R}$	$(0, \pi)$
$y = \sec^{-1}x \Leftrightarrow x = \sec y$	$x \le -1 \wedge x \geq 1$	$[0, \pi] - \{\pi/2\}$
$y = \operatorname{cosec}^{-1}x \Leftrightarrow x = \operatorname{cosec} y$	$x \le -1 \wedge x \geq 1$	$[-\frac{\pi}{2}, \frac{\pi}{2}] - \{0\}$

Remarks:

* The inverse trigonometric functions are defined on principal domains "the domain inwhich functions are one-one".
* In case of inverse trigonometric functions:

$$\sin^{-1}x \neq (\sin x)^{-1} \quad ; \quad \operatorname{cosec}^{-1}x \neq (\operatorname{cosec} x)^{-1}$$
$$\cos^{-1}x \neq (\cos x)^{-1} \quad ; \quad \sec^{-1}x \neq (\sec x)^{-1}$$
$$\tan^{-1}x \neq (\tan x)^{-1} \quad ; \quad \cot^{-1}x \neq (\cot x)^{-1}$$

* Inverse trigonometric functions exist only if trigonometric functions are one-one.
* The graphs of trigonometric functions are along X-axis while graphs of inverse trigonometric functions are along Y-axis.
* The inverse trigonometric functions have wide uses in the Higher Mathematics, Geometry, Trigonometry and Calculus.

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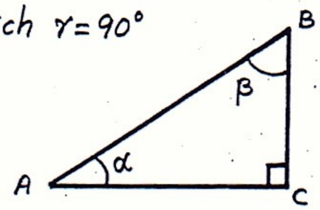
Inverse Trigonometric Identities:

Consider a right triangle ABC inwhich $\gamma = 90^\circ$



then $\alpha + \beta + \gamma = 180^\circ$

$\Rightarrow \alpha + \beta = 180^\circ - 90^\circ = 90^\circ$

$\Rightarrow \alpha = 90^\circ - \beta = \frac{\pi}{2} - \beta$

Now $\sin\alpha = \sin(90^\circ - \beta)$

$\Rightarrow \sin\alpha = \cos\beta \qquad \because \sin(90^\circ - \theta) = \cos\theta$

let $\sin\alpha = \cos\beta = x$

$\Rightarrow \sin\alpha = x$ and $\cos\beta = x$.

$\Rightarrow \alpha = \sin^{-1}x$ and $\beta = \cos^{-1}x$

so $\boxed{\sin^{-1}x = \frac{\pi}{2} - \cos^{-1}x}$ (Proved)

or $\boxed{\cos^{-1}x = \frac{\pi}{2} - \sin^{-1}x}$

Again $\tan\alpha = \tan\left(\frac{\pi}{2} - \beta\right)$ as $\alpha = \frac{\pi}{2} - \beta$

$\Rightarrow \tan\alpha = \cot\beta \qquad \because \cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$

Let $\tan\alpha = \cot\beta = x$

$\Rightarrow \tan\alpha = x$ and $\cot\beta = x$

$\Rightarrow \alpha = \tan^{-1}x$ and $\beta = \cot^{-1}x$

so $\boxed{\tan^{-1}x = \frac{\pi}{2} - \cot^{-1}x}$

or $\boxed{\cot^{-1}x = \frac{\pi}{2} - \tan^{-1}x}$ (Proved)

Again $\alpha = \frac{\pi}{2} - \beta$

$\Rightarrow \sec\alpha = \sec\left(\frac{\pi}{2} - \beta\right)$

$\Rightarrow \sec\alpha = \operatorname{cosec}\beta \qquad \because \operatorname{cosec}\theta = \sec\left(\frac{\pi}{2} - \theta\right)$

Let $\sec\alpha = \operatorname{cosec}\beta = x$

$\Rightarrow \sec\alpha = x$ and $\operatorname{cosec}\beta = x$

$\alpha = \sec^{-1}x$ and $\beta = \operatorname{cosec}^{-1}x$

so $\boxed{\sec^{-1}x = \frac{\pi}{2} - \operatorname{cosec}^{-1}x}$ (Proved) $\boxed{\operatorname{cosec}^{-1}x = \frac{\pi}{2} - \sec^{-1}x}$ (Proved)

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Exercise : 13.1

Q.1 Evaluate the followings:

(i) $\sin^{-1}(1)$

Let $y = \sin^{-1}(1)$ where $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\Rightarrow \sin y = 1$

$\because \sin\frac{\pi}{2} = 1$

So $y = \frac{\pi}{2}$

$\Rightarrow \sin^{-1}(1) = \frac{\pi}{2}$ Ans.

(ii) $\sin^{-1}(-1)$

Let $y = \sin^{-1}(-1)$ where $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\Rightarrow \sin y = -1$

$\because \sin\left(-\frac{\pi}{2}\right) = -1$

So $y = -\frac{\pi}{2}$

$\Rightarrow \sin^{-1}(-1) = -\frac{\pi}{2}$ Ans.

(iii) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

Let $y = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ where $y \in [0, \pi]$

$\Rightarrow \cos y = \frac{\sqrt{3}}{2}$

$\because \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

So $y = \frac{\pi}{6}$

$\Rightarrow \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$ Ans.

(iv) $\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

Let $y = \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$ $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\Rightarrow \tan y = \frac{-1}{\sqrt{3}}$

$\because \tan\left(-\frac{\pi}{6}\right) = \frac{-1}{\sqrt{3}}$

So $y = -\frac{\pi}{6}$

$\Rightarrow \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) = \frac{-\pi}{6}$ Ans.

(v) $\cos^{-1}(1/2)$

Let $y = \cos^{-1}(1/2)$, $y \in [0, \pi]$

$\Rightarrow \cos y = 1/2$

$\because \cos\frac{\pi}{3} = \frac{1}{2}$

So $y = \pi/3$

$\Rightarrow \cos^{-1}(1/2) = \pi/3$ Ans.

(vi) $\tan^{-1}(1/\sqrt{3})$

Let $y = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$; $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\Rightarrow \tan y = \frac{1}{\sqrt{3}}$

$\because \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$

So $y = \pi/6$

$\Rightarrow \tan^{-1}(1/\sqrt{3}) = \pi/6$ Ans.

(vii) $\cot^{-1}(-1)$

Let $y = \cot^{-1}(-1)$; $y \in (0, \pi)$

$\Rightarrow \cot y = -1$

$\because \cot\left(\frac{3\pi}{4}\right) = -1$

So $y = \frac{3\pi}{4}$

$\Rightarrow \cot^{-1}(-1) = \frac{3\pi}{4}$ Ans.

(viii) $\operatorname{cosec}^{-1}\left(\frac{-2}{\sqrt{3}}\right)$

Let $y = \operatorname{cosec}^{-1}\left(\frac{-2}{\sqrt{3}}\right)$; $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

$\Rightarrow \operatorname{cosec} y = \frac{-2}{\sqrt{3}}$

$\because \operatorname{cosec}\left(-\frac{\pi}{3}\right) = -\frac{2}{\sqrt{3}}$

So $y = -\frac{\pi}{3}$

$\Rightarrow \operatorname{cosec}^{-1}\left(\frac{-2}{\sqrt{3}}\right) = -\frac{\pi}{3}$ Ans.

(ix) $\sin^{-1}\left(\frac{-1}{\sqrt{2}}\right)$

Let $y = \sin^{-1}(-1/\sqrt{2})$ where $y \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

$\Rightarrow \sin y = \frac{-1}{\sqrt{2}}$

$\because \sin\left(\frac{-\pi}{4}\right) = \frac{-1}{\sqrt{2}}$

So $y = -\pi/4$

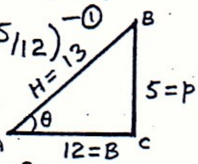
$\Rightarrow \sin^{-1}(-1/\sqrt{2}) = \frac{-\pi}{4}$ Ans.

Q.2 Show that:

(i) $\tan^{-1}(5/12) = \sin^{-1}(5/13)$

Let $\theta = \tan^{-1}(5/12)$ — ①

$\Rightarrow \tan\theta = 5/12$



$\because P = 5 \quad B = 12 \quad H = ?$

$\because H^2 = P^2 + B^2 = 25 + 144 = 169$

$H = 13$

So $\sin\theta = \frac{P}{H} = \frac{5}{13}$

$\theta = \sin^{-1}(5/13)$ — ②

①,② $\Rightarrow \tan^{-1}\left(\frac{5}{12}\right) = \sin^{-1}\left(\frac{5}{13}\right)$ (Proved)

(ii) $2\cos^{-1}(4/5) = \sin^{-1}\left(\frac{24}{25}\right)$

Let $\theta = \cos^{-1}(4/5)$ — ①

$\cos\theta = 4/5$

$\sin\theta = \sqrt{1 - \cos^2\theta}$

$= \sqrt{1 - \frac{16}{25}} = \sqrt{\frac{25-16}{25}}$

$= \sqrt{\frac{9}{25}} = \frac{3}{5}$

Consider

$\sin 2\theta = 2\sin\theta\cos\theta$

$\sin 2\theta = 2 \times \frac{3}{5} \times \frac{4}{5} = \frac{24}{25}$

$2\theta = \sin^{-1}\left(\frac{24}{25}\right)$ — ②

①,② $\Rightarrow 2\cos^{-1}(4/5) = \sin^{-1}\left(\frac{24}{25}\right)$ (Proved)

(iii) $\cos^{-1}(4/5) = \cot^{-1}\left(\frac{4}{3}\right)$

Let $\theta = \cos^{-1}(4/5)$ — ①

$\cos\theta = 4/5$

$\sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{1 - \frac{16}{25}}$

$\sin\theta = \sqrt{\frac{25-16}{25}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$

Now

$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{4/5}{3/5} = \frac{4}{3}$

$\theta = \cot^{-1}(4/3)$ — ②

①,② $\Rightarrow \cos^{-1}(4/5) = \cot^{-1}(4/3)$ (Proved)

Example:- Show that:

$\cos^{-1}\left(\frac{12}{13}\right) = \sin^{-1}\left(\frac{5}{13}\right)$

Proof:- Let $\theta = \cos^{-1}\left(\frac{12}{13}\right)$ — ①

$\Rightarrow \cos\theta = \frac{12}{13}$

$\because \sin\theta = \sqrt{1 - \cos^2\theta}$

$\sin\theta = \sqrt{1 - \frac{144}{169}}$

$\sin\theta = \sqrt{\frac{169-144}{169}} = \sqrt{\frac{25}{169}}$

$\sin\theta = \frac{5}{13}$

$\theta = \sin^{-1}\left(\frac{5}{13}\right)$ — ②

①,② $\Rightarrow \cos^{-1}\left(\frac{12}{13}\right) = \sin^{-1}\left(\frac{5}{13}\right)$ (Proved)

$\cos^{-1}x + \sin^{-1}x = \frac{\pi}{2}$ $\quad$ $\sin^{-1}x = \cos^{-1}\sqrt{1-x^2}$

$\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$ $\quad$ $\sin^{-1}x = \operatorname{cosec}^{-1}(1/x)$

$\sec^{-1}x + \operatorname{cosec}^{-1}x = \frac{\pi}{2}$ $\quad$ $\cos^{-1}x = \sec^{-1}(1/x)$

$\cos[\cos^{-1}x] = x$ $\quad$ $\tan^{-1}x = \cot^{-1}(1/x)$

$\tan[\tan^{-1}x] = x$ $\quad$ $\sin[\sin^{-1}x] = x$

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Q.3 Find the value of:

(i) $\cos\left[\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)\right]$

Let $\theta = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$; $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\sin\theta = \frac{1}{\sqrt{2}}$

$\Rightarrow \theta = \pi/4$

Now

$\cos\left[\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)\right] = \cos\theta$

$\cos\left[\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)\right] = \cos\frac{\pi}{4}$

$\cos\left[\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)\right] = \frac{1}{\sqrt{2}}$

(ii) $\sec\left[\cos^{-1}\left(\frac{1}{2}\right)\right]$

Let $y = \cos^{-1}\left(\frac{1}{2}\right)$ $\quad y \in [0, \pi]$

$\cos y = \frac{1}{2}$

$\Rightarrow y = \frac{\pi}{3}$

Now

$\sec\left[\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)\right] = \sec y$

$\sec\left[\cos^{-1}\left(\frac{1}{2}\right)\right] = \sec\frac{\pi}{3}$

$\sec\left[\cos^{-1}\left(\frac{1}{2}\right)\right] = \frac{1}{\cos\pi/3} = \frac{1}{1/2}$

$\sec\left[\cos^{-1}\left(\frac{1}{2}\right)\right] = 2$

(iii) $\tan\left[\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)\right]$

Let $y = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$, $y \in [0, \pi]$

$\cos y = \frac{\sqrt{3}}{2}$

$\Rightarrow y = \frac{\pi}{6}$

Now

$\tan\left[\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)\right] = \tan y$

$\tan\left[\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)\right] = \tan\frac{\pi}{6}$

$\tan\left[\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)\right] = \frac{1}{\sqrt{3}}$

$\cot\left[\cot^{-1}x\right] = x$

$\sec\left[\sec^{-1}x\right] = x$

$\operatorname{cosec}\left[\operatorname{cosec}^{-1}x\right] = x$

(iv) $\operatorname{cosec}\left[\tan^{-1}(-1)\right]$

Let $y = \tan^{-1}(-1)$ $\quad y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\tan y = -1$

$y = -\frac{\pi}{4}$

Now

$\operatorname{cosec}\left[\tan^{-1}(-1)\right] = \operatorname{cosec}\left(-\frac{\pi}{4}\right)$

$\operatorname{cosec}\left[\tan^{-1}(-1)\right] = -\operatorname{cosec}\pi/4$

$\operatorname{cosec}\left[\tan^{-1}(-1)\right] = -\sqrt{2}$

(v) $\sec\left[\sin^{-1}\left(\frac{-1}{2}\right)\right]$

Let $y = \sin^{-1}\left(-\frac{1}{2}\right)$ $\quad y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\sin y = -\frac{1}{2}$

$y = -\pi/6$

Now

$\sec\left[\sin^{-1}\left(\frac{-1}{2}\right)\right] = \sec\left(-\frac{\pi}{6}\right)$

$\sec\left[\sin^{-1}\left(\frac{-1}{2}\right)\right] = \sec\pi/6$

$\sec\left[\sin^{-1}\left(\frac{-1}{2}\right)\right] = \frac{2}{\sqrt{3}}$

(vi) $\tan(\tan^{-1}(-1))$

$\because$ tan and $\tan^{-1}$ are inverses so

$\tan\left[\tan^{-1}(-1)\right] = -1$

(vii) $\sin\left[\sin^{-1}\left(\frac{1}{2}\right)\right]$

$\because$ Sin and $\sin^{-1}$ are inverses so

$\sin\left[\sin^{-1}\left(\frac{1}{2}\right)\right] = \frac{1}{2}$

$\because$ $\sin^{-1}(-x) = -\sin^{-1}x$

$\cos^{-1}(-x) = \pi - \cos^{-1}x$

$\tan^{-1}(-x) = -\tan^{-1}x$

TAHIR MEHMOOD
M.Sc Math
0345-6510779

viii) $\tan\left[\sin^{-1}\left(\frac{-1}{2}\right)\right]$

Let $y = \sin^{-1}\left(\frac{-1}{2}\right)$, $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\sin y = \frac{-1}{2}$

$y = -\frac{\pi}{6}$

Now

$\tan\left[\sin^{-1}\left(\frac{-1}{2}\right)\right] = \tan\left(-\frac{\pi}{6}\right)$

$\tan\left[\sin^{-1}\left(\frac{-1}{2}\right)\right] = -\tan\frac{\pi}{6}$

$\tan\left[\sin^{-1}\left(\frac{-1}{2}\right)\right] = -\frac{1}{\sqrt{3}}$

(ix) $\sin\left[\tan^{-1}(-1)\right]$

Let $y = \tan^{-1}(-1)$, $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\tan y = -1$

$y = -\frac{\pi}{4}$

Now

$\sin\left[\tan^{-1}(-1)\right] = \sin\left(-\frac{\pi}{4}\right)$

$\sin\left[\tan^{-1}(-1)\right] = -\sin\frac{\pi}{4}$

$\sin\left[\tan^{-1}(-1)\right] = -\frac{1}{\sqrt{2}}$

Prove the Formulae:-

(i) $\sin^{-1}A + \sin^{-1}B = \sin^{-1}\left(A\sqrt{1-B^2} + B\sqrt{1-A^2}\right)$

(ii) $\sin^{-1}A - \sin^{-1}B = \sin^{-1}\left(A\sqrt{1-B^2} - B\sqrt{1-A^2}\right)$

Proof:- Let $\alpha = \sin^{-1}A$ & $\beta = \sin^{-1}B$

$\sin\alpha = A$ $\qquad \sin\beta = B$

$\cos\alpha = \sqrt{1-\sin^2\alpha}$, $\cos\beta = \sqrt{1-\sin^2\beta}$

$= \sqrt{1-A^2}$, $= \sqrt{1-B^2}$

Now

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\sin(\alpha+\beta) = A\sqrt{1-B^2} + B\sqrt{1-A^2}$

$\alpha+\beta = \sin^{-1}\left(A\sqrt{1-B^2} + B\sqrt{1-A^2}\right)$

Putting α, β, we have

$$\boxed{\sin^{-1}A + \sin^{-1}B = \sin^{-1}\left(A\sqrt{1-B^2} + B\sqrt{1-A^2}\right)}$$

Now

$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$\sin(\alpha-\beta) = A\sqrt{1-B^2} - B\sqrt{1-A^2}$

$\alpha-\beta = \sin^{-1}\left(A\sqrt{1-B^2} - B\sqrt{1-A^2}\right)$

Putting α, β, we have

$$\boxed{\sin^{-1}A - \sin^{-1}B = \sin^{-1}\left(A\sqrt{1-B^2} - B\sqrt{1-A^2}\right)}$$

(iii) $\cos^{-1}A + \cos^{-1}B = \cos^{-1}\left(AB - \sqrt{(1-A^2)(1-B^2)}\right)$

(iv) $\cos^{-1}A - \cos^{-1}B = \cos^{-1}\left(AB + \sqrt{(1-A^2)(1-B^2)}\right)$

Proof:- Let $\alpha = \cos^{-1}A$ & $\beta = \cos^{-1}B$

$\cos\alpha = A$ $\qquad \cos\beta = B$

$\sin\alpha = \sqrt{1-\cos^2\alpha}$, $\sin\beta = \sqrt{1-\cos^2\beta}$

$\sin\alpha = \sqrt{1-A^2}$, $\sin\beta = \sqrt{1-B^2}$

Now

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\cos(\alpha+\beta) = AB - \sqrt{1-A^2}\cdot\sqrt{1-B^2}$

$\alpha+\beta = \cos^{-1}\left(AB - \sqrt{(1-A^2)(1-B^2)}\right)$

Putting α, β, we have

$$\boxed{\cos^{-1}A + \cos^{-1}B = \cos^{-1}\left(AB - \sqrt{(1-A^2)(1-B^2)}\right)}$$

Now

$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$\cos(\alpha-\beta) = AB + \sqrt{1-A^2}\cdot\sqrt{1-B^2}$

$\alpha-\beta = \cos^{-1}\left(AB + \sqrt{(1-A^2)(1-B^2)}\right)$

Putting α, β, we have

$$\boxed{\cos^{-1}A - \cos^{-1}B = \cos^{-1}\left(AB + \sqrt{(1-A^2)(1-B^2)}\right)}$$