Chapter 12

Laplace Transformation

12.1 Introduction:

We now present a systematic and elegant procedure that is widely used in circuit analysis and in the study of feedback and control. The theory takes its form from a symbolic method developed by the English engineer Oliver Heaviside. The modern approach to this method is based on the Laplace transformation. It enables one to solve many problems without going to the trouble of finding the general solution and then evaluating the arbitrary constants. The procedure can be extended to systems of equations to partial differential equations of electrical network and mechanical oscillations and to integral equations and it often yields results more readily than other techniques.

12.2 Laplace transformation:

The basis of this method is the transformation defined by

$$\overline{f(s)} = F(s) = \int_0^\infty f(t)\, e^{-st}\, dt = L\ \{f(t)\} \ldots\ldots\ldots\ldots\ldots(1)$$

The function F(s) is the Laplace transform of f(t), and the operator L that transforms f(t) into F(s) is the Laplace transform operator. The functional relation expressed between F(s) and f(t) is written in the form F(s) = L {f(t)}.

It should be emphasized that equation (1) describes the action of L, not only on f(t) but on any function to which L can be applied Thus.

$$L\ \{U(t)\} = \int_0^\infty U(t)\, e^{-st} dt\ , \quad L\ \{V(t)\} = \int_0^\infty V(t)\, e^{-st}\, dt \text{ and so on}$$

Example 1:

Let $f(t) = 1$ when $t \geq 0$, Find $\overline{f(s)}$ (Laplace transform)

Solution: Since $f(t) = 1$

$$\therefore \quad \overline{f(s)} = L\ \{f(t)\} = \int_0^\infty f(t)\, e^{-st}.\, dt$$

$$\overline{f(s)} = L\{f(t)\} = \int_0^\infty 1.e^{-st}\,dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= -\frac{1}{s}\left[e^{-\infty} - e^{0}\right]$$

$$= -\frac{1}{s}[0-1] \qquad \therefore \qquad e^{-\infty} = 0$$

$$= \frac{1}{s}$$

Example 2:

Let f(t) = t when $t \geq 0$, Find L{f(t)}

Solution : Since $f(t) = t$

$$\therefore \qquad \overline{f(s)} = L\{f(t)\} = \int_0^\infty f(t)\, e^{-st}.dt$$

$$\overline{f(s)} = L\{f(t)\} = \int_0^\infty t.e^{-st}\,dt$$

Integration by parts

$$= \left[t.\frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}\,dt$$

$$= 0 + \frac{1}{s}\int_0^\infty e^{-st}.\,dt$$

$$= \frac{1}{s}\left[\frac{e^{-st}}{-s}\right]_0^\infty = -\frac{1}{s^2}\left[e^{-\infty} - e^{0}\right]$$

$$= -\frac{1}{s^2}[0-1] = \frac{1}{s^2}$$

Example 3:

Let $f(t) = e^{at}$ When $t \geq 0$, and a is constant, Find L{f(t)}

Solution: Since $f(t) = e^{at}$

$\therefore \quad \overline{f(s)} = L\{f(t)\} = \int_0^\infty f(t)\, e^{-st}.\, dt$

$$L\{e^{at}\} = \int_0^\infty e^{-st}.\, e^{at}\, dt = \int_0^\infty e^{-(S-a).t}\, dt$$

$$= \left| \frac{e^{-(s-a).t}}{-(s-a)} \right|_0^\infty$$

$$= -\frac{1}{s-a}\left[e^{-\infty} - e^{0}\right] = -\frac{1}{s-a}[0-1]$$

$$= \frac{1}{s-a} \quad ; \quad s > a$$

12.3 Properties of Laplace Transformation :

Linearity :

1. $$L\{c\, f(t)\} = \int_0^\infty c f(t)\, e^{-st}\, dt$$

$$= c\int_0^\infty f(t)\, e^{-st}\, dt$$

$$L\{c\, f(t)\} = c\, L\{f(t)\}$$

2. $$L\{\alpha f(t) + \beta\, g(t)\} = \int_0^\infty \{\alpha f(t) + \beta\, g(t)\, e^{-st}\, dt\}$$

$$= \int_0^\infty \alpha f(t) e^{-st}\, dt + \int_0^\infty \beta\, g(t)\, e^{-st}\, dt$$

$$= \int_0^\infty \alpha f(t) e^{-st}\, dt + \beta \int_0^\infty g(t)\, e^{-st}\, dt$$

$$L\{\alpha\, f(t) + \beta\, g(t) = \alpha L\, f(t) + \beta\, L\, g(t)$$

Which shows that the operator L is linear.

Laplace transform of derivatives :

3. $$L[f'(t)] = \int_0^\infty e^{-st}.\, f'(t)\, dt$$

Integrating by parts

$$= \left|(e^{-st}) f(t)\right|_0^\infty + s\int_0^\infty e^{-st}. f(t)\,dt$$

$$= [e^{-\infty} f(\infty) - e^0 f(0)] + s\,L f(t)$$

$$= -f(0) + s\,L\,(f(t)$$

$$L[f'(t)] = s\,L\,f(t) - f(0)$$

4. $L[f''(t)] = s\,L\,[f'(t)] - f'(0)$

$$= s\,[s\,Lf(t)] - f(0)] - f'(0)$$

$$L[f''(t)] = s^2 L\,[f(t)] - s\,f(0) - f'(0)$$

5. $L[f'''(t)] = s^2 L\,[f'(t)] - s\,f'(0) - f''(0)$

$$= s^2[s\,L\,f(t) - f(0)] - s\,f'(0) - f''(0)$$

$$= s^3\,L\,f(t) - s^2 f(0) - s\,f'(0) - f''(0)$$

Repetition of this process gives.

$$L[f^m(t)] = s^m L\,f(t) - s^{m-1} f(0) - s^{m-2} f_1(0) \ldots - s\,f_{m-2}(0) - f_{m-1}(0)$$

Where , $f_1(0) = f'(0), \ldots\ldots, f_{m-1}(0) = f^{(m-1)}(0)$

Example 4: Find $L(t^n)$

Solution: Let $f(t) = t^n \Rightarrow f(0) = 0$

$f_1(t) = n\,t^{n-1} \Rightarrow f_1(0) = 0$

$f_2(t) = n(n-1)\,t^{n-2} \Rightarrow f_2(0) = 0$

$f_{n-1}(t) = n(n-2)\ldots\ldots\ 2.t \Rightarrow f_{n-1}(0) = 0$

$f_n(t) = f^n(t) = n\,(n-1)\,(n-2)\ldots\ldots 2.1 = n!$

By property 4

$$L\,\{f^n(t)\} = S^n L\,f(t) - S^{n-1} f(0) - S^{n-2} f_1(0) - \ldots\ldots f_{n-1}(0)$$

$$L\{n\,!\} = S^n L\,(t^n)$$

$$n!\,L\{1\} = S^n L\,(t^n)$$

$$n!\left(\frac{1}{s}\right) = S^n L\,(t^n)$$

$$L(t^n) = \frac{n!}{s^{n+1}}$$

Example 5:

Find $L(t^3)$

Solution: **Method – I**

Since $L(t^n) = \frac{n!}{s^{n+1}}$

Since $L(t^3) = \frac{3!}{s^{3+1}} = \frac{6}{S^4}$

Method – 2:

Let $f(t) = t^3$, then $f(0) = 0$

$f'(t) = 3t^2$, $f(0) = 0$

$f''(t) = 6t$, $f''(0) = 0$

$f'''(t) = 6$

By the formula (property – 5)

$Lf'''(t) = s^3 L f(t) - s^2 f(0) - s f'(0) - f''(0)$

$L\{6\} = s^3 L f(t) - s^2(0) - s(0) - 0$

$6L\{1\} = s^3 L f(t)$

$6\left(\frac{1}{s}\right) = s^3 L(f(t))$

$\therefore \quad L(t^3) = \frac{6}{s^4}$

Example 6:

(i) Find $L[\sin \omega t]$ (ii) $L[\cos \omega t]$

Solution: **Method –I**

Since by example 2

$L(e^{at}) = \frac{1}{S-a}$

Put $a = i\omega$

$L(e^{i\omega t}) = \frac{1}{S - i\omega}$

$L(\cos \omega t + i \sin \omega t) = \frac{1}{S - i\omega} \times \frac{S + i\omega}{S + i\omega} =$

$$L(\cos \omega t) + i\, L(\sin \omega t) = \frac{S}{S^2+\omega^2} + i\,\frac{\omega}{S^2+\omega^2}$$

Comparing the real and imaginary parts we get

(i) $L[\cos \omega t] = \dfrac{S}{S^2+\omega^2}$ (Real part)

(ii) $L[\sin \omega t] = \dfrac{\omega}{S^2+\omega^2}$ (Imaginary part)

Method – 2: (i) Find $L[\sin \omega t]$

Here, $f(t) = \sin \omega t$, $f(0) = \sin 0 = 0$

$f'(t) = \omega \cos \omega t$, $f'(0) = \omega \cos 0 = \omega$

$f''(t) = -\omega^2 \sin \omega t$

Since,

$$L[f''(t)] = S^2 L[f(t)] - S f(0) - f'(0)$$

$$L[-\omega^2 \sin \omega t] = S^2 L[\sin \omega t] - 0 - \omega$$

$$-\omega^2 L[\sin \omega t] - S^2 L[\sin \omega t] = -\omega$$

$$(S^2+\omega^2) L[\sin \omega t] = \omega$$

$$L[\sin \omega t] = \frac{\omega}{S^2+\omega^2}$$

(ii) Find $L[\cos \omega t]$

Here, $f(t) = \cos \omega t$, $f(0) = \cos 0 = 1$

$f'(t) = -\omega \sin \omega t$, $f'(0) = -\omega \sin 0 = 0$

$f''(t) = -\omega^2 \cos \omega t$

Since,

$$L[f''(t)] = S^2 L[f(t)] - S f(0) - f'(0)$$

$$L[f''(t)] = S^2 L[\cos \omega t] - S f(0) - f'(0)$$

$$L[-\omega^2 \cos \omega t] = S^2 L[\cos \omega t] - S - 0$$

$$(S^2+\omega^2) L[\cos \omega t] = S$$

$$L[\cos \omega t] = \frac{S}{S^2+\omega^2}$$

Property 6: Laplace transform of the derivative of order n :

Let f(t) be continuous function for $t \geq 0$, then the Laplace transforms of nth order derivative is

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}[L\{f(t)\}]$$

Proof:

Since $\overline{f(s)} = L\{f(t)\} = \int_0^\infty e^{-st}. f(t)\, dt$

Differentiating w.r.t. s

$$\frac{d}{ds}\left[\overline{f(s)}\right] = \int_0^\infty -t.\, e^{-st}.\, f(t)\, dt$$

$$\frac{d}{ds}\left[\overline{f(s)}\right] = (-1)\int_0^\infty t.\, e^{-st}.\, f(t)\, dt$$

$$(-1)\frac{d}{ds}\left[\overline{f(s)}\right] = L\{t\, f(t)\}$$

On differentiating once again

$$(-1)\frac{d^2}{ds^2}\left[\overline{f(s)}\right] = \int_0^\infty t.e^{-st} f(t)\,(-t)$$

$$(-1)\frac{d^2}{ds^2}\left[\overline{f(s)}\right] = (-1)\int_0^\infty t^2\, e^{-st} f(t)\, dt$$

$$\therefore \quad (-1)^2\frac{d^2}{ds^2}\left[\overline{f(s)}\right] = L\{t^2 f(t)\}$$

Similarly

$$(-1)^n\frac{d^n}{ds^n}\left[\overline{f(s)}\right] = L\{t^n f(t)\}$$

Example 7:

Let $f(t) = t^n$ for all $t \geq 0$, find $L\{t^n\}$

Solution:

$$L\{t^n f(t)\} = (-1)^n\frac{d^n}{ds^n}\left[\overline{f(s)}\right] = (-1)^n\frac{d^n}{ds^n}[L\{f(t)\}]$$

Here , $f(t) = 1$

$$L\{t^n.1\} = (-1)^n \frac{d^n}{ds^n}\left[\frac{1}{s}\right] \quad \therefore \quad L(1) = \frac{1}{s}$$

$$= (-1)^n \frac{(-1)^n\, n!}{s^{n+1}}$$

$$= \frac{n!}{s^{n+1}}$$

12.4 Inverse Laplace Transforms:

The operation by which we recover f(t) from L{f(t)} = F(s) is called inverse Laplace Transform and is denoted by L^{-1}.

Thus if L{f(t)} = F (s)

Then f(t) = L^{-1} [F(s)]

Table of Laplace Transforms and Inverse Laplace Transforms

Sr#	f(t)	L f(t)	L {f (t)}	f(t) = L^{-1} [F(s)]
1	1	$^1/_s$	$^1/_s$	1
2	t	$^1/_{s^2}$	$^1/_{s^2}$	t
3	t^n	$\frac{n!}{S^{n+1}}$	$\frac{n!}{S^{n+1}}$	t^n
4	e^{ct}	$\frac{1}{S-C}$	$\frac{1}{S-C}$	e^{ct}
5	e^{-ct}	$\frac{1}{S+C}$	$\frac{1}{S+C}$	e^{-ct}
6	sin ωt	$\frac{\omega}{S^2+\omega^2}$	$\frac{\omega}{S^2+\omega^2}$	sin ωt
7	cos ω t	$\frac{S}{S^2+\omega^2}$	$\frac{S}{S^2+\omega^2}$	cos ωt
8	$t\,e^{-ct}$	$\frac{1}{(S+C)^2}$	$\frac{1}{(S+C)^2}$	$t\,e^{-ct}$
9	$\frac{t^{n-1}\,e^{-ct}}{(n-1)!}$ n is + ve	$\frac{1}{(S+C)^n}$	$\frac{1}{(S+C)^n}$	$\frac{t^{n-1}\,e^{-ct}}{(n-1)!}$
10	$e^{-ct}(1-ct)$	$\frac{S}{(S+C)^2}$	$\frac{S}{(S+C)^2}$	$e^{-ct}(1-ct)$

Example 8:

Find $L^{-1}[f(s)]$ of $\frac{s}{(s-a)(s-b)}$, $a \neq b$

Solution:

$$L^{-1}\left\{\frac{s}{(s-a)(s-b)}\right\} = L^{-1}\left[\frac{1}{a-b}\left\{\frac{a}{s-a} - \frac{b}{s-b}\right\}\right] \text{ (by partial fraction)}$$

$$= \frac{1}{a-b}\left[L^{-1}\left(\frac{a}{s-a}\right) - L^{-1}\left(\frac{b}{s-b}\right)\right]$$

$$= \frac{1}{a-b}\left(a\,e^{at} - b\,e^{bt}\right)$$

Note:

$$\therefore \quad L^{-1}\left(\frac{1}{S-a}\right) = e^{at}$$

$$L^{-1}\left(\frac{1}{s-b}\right) = e^{bt}$$

Exercise 12

1. Find the Laplace transforms of the following functions.
   (i) 4 (ii) $3t$ (iii) t^2
   (iv) at (v) $t^2 - 2t$ (vi) e^{-at}
2. Find the Laplace transforms of the following functions.
   (i) $3t + 4$ (ii) $t^2 + at + b$ (iii) $\cos 3t$
   (iv) $\sin 4t$ (v) $a \cos 2t$ (vi) $\sin t \,.\, \cos t$
3. Show that Laplace transforms of

$$L\{t^2 f'(t)\} = S\overline{f''(s)} + 2\overline{f'(s)}$$

4. Prove that

   (i) $L\{e^{at}\cos \omega t\} = \frac{s-a}{(s-a)^2 + \omega^2}$

   (ii) $L\{e^{at}\sin \omega t\} = \frac{\omega}{(s-a)^2 + \omega^2}$

5. Let $f(t) = 2 \sin \omega t$. Find $L\{f(t)\}$

6. Find inverse Laplace transforms of the following.

(i) $\frac{5}{s-3}$ (ii) $\frac{1}{s^2+25}$ (iii) $\frac{9}{s^2+3S}$

(iv) $\frac{1}{s(s+1)}$ (v) $\frac{1}{s(s^2+1)}$ (vi) $\frac{1}{(s+1)(s-2)}$

(vii) $\frac{1}{(s+a)(s+b)}$

Answers

Q.1: (i) $\frac{4}{s}$ (ii) $\frac{3}{s^2}$ (iii) $\frac{2!}{s^2}$

(iv) $\frac{a}{s^2}$ (v) $\frac{2!}{s^3} - \frac{2}{s^2}$ (vi) $\frac{1}{s+a}$

Q.2: (i) $\frac{3}{S^2}+\frac{4}{S}$ (ii) $\frac{2}{s^3}+\frac{a}{s^2}+\frac{b}{s}$ (iii) $\frac{s}{s^2+3^2}$

(iv) $\frac{4}{s^2+4^2}$ (v) $\frac{a\,s}{s^2+2^2}$ (vi) $\frac{1}{s^2+4}$

Q.5: $\frac{2\omega}{s^2+\omega^2}$

Q.6: (i) $5e^{3t}$ (ii) $\frac{1}{5}\sin 5t$ (iii) $3-3e^{3t}$

(iv) $1-e^{3t}$ (v) $1-\cos t$ (vi) $\frac{1}{2}[e^{-t} - e^{2t}]$

(vii) $\frac{1}{(a-b)}[-e^{-at}+e^{-bt}]$