Chapter 11

Fourier Series

11.1 Introduction:

It was J.B. Fourier who used such expansions in different types of problems. A specific infinite trigonometric series used by him was of the form.

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos n x + b_n \sin nx)$$

As trigonometric functions used in this series are periodic, therefore this series is a periodic function of period 2π.

This series is of fundamental importance in the study of physical systems subjected to periodic disturbances. It has waste applications in the present scientific age. For example, the voltage impressed on an electrical circuit might consist of a series of pulses or the disturbing influence acting on mechanical system might be a force of constant magnitude whose direction is periodically and instantaneously reversed. This theory is also applicable in many types of engineering problems. Its most important application is in the analysis of the behavior of physical systems subjected to periodic disturbances.

Before going to Fourier Series, some important terms are being explained.

11.2 Periodic Function:

If we add or substract a constant from the argument of a function, it remains with no change or we say $f(x) = f(x \pm A) \; \forall \; x$

Here A is called period of the function. Period of sin x and cos x is same, which is 2π.

11.3 Even Functions:

If for a function $f(x)$

$f(-x) = f(x)$

Then f(x) is called an even function.

For example, $f(x) = x^2$

$f(-x) = (-x)^2 = x^2 = f(x)$

And for $f(x) = \cos x$

$f(-x) = \cos(-x) = \cos x = f(x)$

Therefore both x^2 and $\cos x$ are even functions of x

11.4 Odd Function:

If for a function $f(x)$

$f(-x) = -f(x)$

Then f (x) is called an odd function, for example

$f(x) = x^3$

$f(-x) = (-x)^3 = -x^3 = -f(x)$

And for $f(x) = \sin x$

$f(-x) = \sin(-x) = -\sin x = -f(x)$

Therefore both x^3 and $\sin x$ are odd functions of x

11.5 Fourier Function:

A trigonometric series of the form

$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Is called Fourier series

Suppose $f(x)$ is a function which can be expressed in the infinite trigonometric series, then

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Here a_0, a_n and b_n are Fourier coefficients of f(x). Now, if $f(x)$ is integrable in the internal $[-\pi, \pi]$. Then Fourier coefficients are defined as

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

and, $$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin n\,x \, dx$$

Now we discuss these coefficients for even and odd functions .

For even function

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx = \frac{2}{\pi} \int_{0}^{\pi} f(x) \, dx$$

for $$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos n\,x \, dx$$

as f(x) is an even function and cos nx is also an even function. Therefore, f(x) cos n x being product of two even functions is also an even function.

Hence $$a_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \cos nx \, dx$$

and for $$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin n\,x \, dx$$

Here f(x) sin nx, being product of an even and an odd function, is odd

Therefore $b_n = 0$ (by definition)

For Odd function

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx = 0 \quad \text{(by definition)}$$

for $$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos n\,x \, dx$$

as f(x) cos n x is an odd function, being product of an odd and an even function.

Therefore

$$a_n = 0 \quad \text{(by definition)}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin n x \, dx$$

$$= \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin n x \, dx$$

Because $f(x) \sin nx$ is an even function, being the product of both odd functions.

Now if we define the periodic function f(x) in the interval $[0, 2\pi]$, then the coefficient of the Fourier series are given by

$$a_0 = \frac{1}{\pi} \int_{0}^{2\pi} f(x) \, dx$$

$$a_n = \frac{1}{\pi} \int_{0}^{2\pi} f(x) \cos n x \, dx$$

and $$b_n = \frac{1}{\pi} \int_{0}^{2\pi} f(x) \sin n x \, dx$$

11.6 Extended Rule of Integration by parts:

This rule is often used in solving the integral in the problems of Fourier Series. This rule of by parts is applied successively a number of times, until we get a standard form of the integrand.

If f and g are two functions, extended rule will be applied in the following form $\int fg \, dx = f g_1 - f' g_2 + f'' g_3 \ldots + (-1)^n f^{n-1} g_n + (-1)^n \int f^n g_n \, dx$

Where f', f'', f''', ...are first, second, third derivatives and so on, and $g_1, g_2, g_3, \ldots\ldots$are first, second, third integrals and so on.

Example 1:

Determine the Fourier series for the following functions.

(i) $f(x) = x^2$, $-\pi \le x \le \pi$

(ii) $f(x) = x^2$, $0 \le x \le 2\pi$

Solution (i):

For the interval $-\pi \le x \le \pi$

We know that x^2 is an even function, therefore $b_n = 0$ and a_0, a_n will be calculated as,

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{\pi}\left(\frac{x^3}{3}\right)_0^{\pi} = \frac{2}{\pi}\frac{\pi^3}{3} = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,dx$$

Here we use extended rule of integration by parts

$$= \frac{2}{\pi}\left[\left| x^2\frac{\sin nx}{n} - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right|_0^{\pi}\right]$$

We know that $\sin 0 = 0$ and $\sin \pi = 0$

Therefore $a_n = \frac{2}{\pi}\left[-2\pi\left(-\frac{\cos n\pi}{n^2}\right) + 2(0)\left(-\frac{\cos n\,0}{n^2}\right)\right]$

$$= \frac{4}{n^2}\operatorname{Cos} n\pi + 0 = \frac{4}{n^2}(-1)^n$$

because for $n = 1, 2, 3 \ldots\ldots\ldots n$, $\Rightarrow \cos n\pi = -1, 1, -1 \ldots\ldots = (-1)^n$

Fourier series for the function is given by

$$x^2 = \frac{1}{2}a_0 + \sum_1^{\infty} a_n\cos nx + b_n\sin nx.$$

$$= \frac{1}{2}\frac{2\pi^2}{3} + \sum_1^{\infty}\left[\frac{4}{n^2}(-1)^n\cos nx + (0)\sin nx\right]$$

$$= \frac{\pi^2}{3} + 4\sum_1^{\infty}(-1)^n\frac{\cos nx}{n^2}$$

$$= \frac{\pi^2}{3} + 4\left(-\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^3} + \ldots\ldots\ldots\ldots\right)$$

Solution (ii):

No we solve $f(x) = x^2$ for $0 \le x \le 2\pi$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2\,dx = \frac{1}{\pi}\left|\frac{x^3}{3}\right|_0^{2\pi}$$

$$= \frac{1}{\pi}\frac{(2\pi)^3}{3} = \frac{8\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x^2\cos nx\,dx$$

Applying extended rule of integration by parts

$$a_n = \frac{2}{\pi}\left[\left| x^2\frac{\sin nx}{n} - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right|_0^{2\pi}\right]$$

Here $\sin 2\pi = \sin 0 = 0$

$$\text{Therefore } a_n = \left[\left\{-2(2\pi)\left(-\frac{\cos 2n\pi}{n^2}\right)\right\} - \left\{-2(0)\left(\frac{\cos 0}{n^2}\right)\right\}\right]$$

$$= \frac{1}{\pi}\left[+\frac{4\pi \cos 2n\pi}{n^2}\right]$$

$$= 4\frac{\cos 2n\pi}{n^2} = \frac{4}{n^2} \text{ because for } n = 1, 2, 3, \ldots \Rightarrow \cos 2n\pi = 1$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x^2 \sin n x \, dx$$

Applying extended rule of integration by parts.

$$b_n = \frac{1}{\pi}\left[x^2\left(-\frac{\cos nx}{n}\right) - (2x)\left(-\frac{\sin nx}{n^2}\right) + (2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[4\pi^2\left(-\frac{\cos 2n\pi}{n}\right) + \frac{2\cos 2n\pi}{n^3}\right] - \frac{1}{\pi}\left[\frac{2\cos 0}{n^3}\right]$$

$$= \frac{-4\pi}{n} + \frac{2}{n^3\pi} - \frac{2}{n^3\pi} = \frac{-4\pi}{n}$$

Now Fourier Series for the given function in the given interval is

$$x^2 = \frac{1}{2}\frac{8\pi^2}{3} + \sum_1^{\infty}\left(\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right)$$

$$= \frac{4\pi^2}{3} + 4\sum_1^{\infty}\left(\frac{\cos nx}{n^2} - \frac{4\pi}{n}\sin n x\right)$$

Example 2: Find the Fourier Series.

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}$$

Solution:

As f(x) is a periodic function of period 2π, therefore its Fourier series is

$$\frac{1}{2} a_0 + \sum_{1}^{\infty}\left(a_n \cos n x + b_n \sin n x\right)$$

So here we evaluate a_0, a_n and b_n

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\, dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} (0)\, dx + \frac{1}{\pi}\int_{0}^{\pi} (1)\, dx$$

$$a_0 = 0 + \frac{1}{\pi}|x|_0^{\pi} = \frac{1}{\pi}(\pi - 0) = 1$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} 0 \cos n x + \frac{1}{\pi}\int_{0}^{\pi} 1 \cos n x\, dx$$

$$= 0 + \frac{1}{\pi}\left[+\frac{\sin n x}{n}\right]_0^{\pi} = \frac{1}{n\pi}[\sin n\pi - \sin 0]$$

$$= \frac{1}{\pi}[0 - 0] = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0} (0) \sin n x\, dx + \frac{1}{\pi}\int_{0}^{\pi} 1 \sin n x\, dx$$

$$= 0 + \frac{1}{\pi}\left[-\frac{\cos n x}{n}\right]_0^{\pi} = -\frac{1}{n\pi}[\cos n\pi - \cos 0]$$

$$= -\frac{1}{n\pi}\left[(-1)^n - 1\right] = \frac{1}{n\pi}\left[(-1)^{n+1} + 1\right]$$

Fourier series becomes

$$f(x) = \frac{1}{2} + \sum_{1}^{\infty}\left[0 \cos n x + \frac{1}{n\pi}\left\{(-1)^{n+1} + 1\right\} \sin n x\right]$$

$$= \frac{1}{2} + \frac{1}{\pi}\left[\frac{2\sin x}{1} + \frac{2\sin 3x}{3} + \frac{2\sin 5x}{5} + \ldots\ldots\ldots\ldots\right]$$

$$= \frac{1}{2} + \frac{2}{\pi}\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\ldots\ldots\ldots\right]$$

Example 3:

Obtain the Fourier Series for e^x in $[-\pi, \pi]$

Solution:

Here a_o, a_n and b_n will be evaluated.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\, dx = \frac{1}{\pi}\left|e^x\right|_{-\pi}^{\pi} = \frac{1}{\pi}(e^{\pi} - e^{-\pi})$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos n x\, dx$$

Let $I_1 = \int e^x \cos n x\, dx = \int \cos nx.\, e^x\, dx$

Let $\cos n x = 1^{st}$ function and $e^x = 2^{nd}$ function

Now we apply integration by parts

$$I_1 = \cos n x \int e^x - \int\left(\frac{d}{dx}\cos nx \int e^x\, dx\right) dx$$

$$= \cos n x\, e^x - \int (-n) \sin n x\, e^x\, dx$$

$$= \cos n x\, e^x + n\left[\sin nx \int e^x\, dx - \int\left(\frac{d}{dx}\sin nx \int e^x\, dx\right) dx\right]$$

$$= \cos n x\, e^x + n\left[\sin nx\, e^x - \int n \cos n x\, e^x\, dx\right]$$

$$I_1 = \cos n x\, e^x + n \sin n x\, e^x - n^2 I_1$$

$$I_1 + n^2 I_1 = (\cos n x + n \sin n x)\, e^x$$

$$I_1(1 + n^2) = (\cos n x + n \sin n x)\, e^x$$

$$I_1 = \frac{e^x}{(1 + n^2)}(\cos n x + n \sin n x)$$

$$a_n = \frac{1}{\pi}\left|\frac{e^x}{(1 + n^2)}(\cos nx + n \sin nx)\right|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{\pi}}{(1+n^2)}\cos n\pi - \frac{e^{-\pi}}{(1+n^2)}\cos(-n\pi)\right] \quad \left\{\begin{array}{l}\therefore \sin nx = 0 \\ \text{for } x = \pi \text{ or } -\pi\end{array}\right.$$

$$= \frac{e^x}{\pi(1+n^2)}\left[e^{\pi}(-1)^n - e^{-\pi}(-1)^n\right]$$

$$a_n = \frac{(-1)^n}{\pi(1+n^2)}\left[e^{\pi} - e^{-\pi}\right]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \, \text{Sin } nx \, dx$$

Let $I_2 = \int \sin n\,x.\, e^x \, dx$

Applying integration by parts by taking

Ist function = sin nx and second functions = e^x

$$I_2 = \sin nx \int e^x dx - \int\left(\frac{d}{dx}\sin nx \int e^x dx\right)dx$$

$$= \sin nx\, e^x - \int n \cos n\,x\, e^x dx$$

$$= \sin nx\, e^x - n\left[\cos nx \int e^x dx - \int\left(\frac{d}{dx}\cos nx \int e^x dx\right)dx\right]$$

$$I_2 = \sin nx\, e^x - n\left[\cos n\,x\, e^x - \int -n \sin n\,x\, e^x dx\right]$$

$$= \sin nx\, e^x - n\cos nx\, e^x - n^2 \int \sin n\,x\, e^x dx]$$

$$I_2 = \sin n\,x\, e^x - n\cos n\,x\, e^x - n^2 I_2$$

$$I_2 + n^2 I_2 = \sin n\,x\, e^x - n\cos n\,x\, e^x$$

$$I_2 = \frac{e^x}{(1+n^2)}(\sin n\,x - n\cos n\,x)$$

$$b_n = \frac{1}{\pi}\left|\frac{e^x}{1+n^2}(\sin nx - n\cos nx)\right|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left\{\frac{ne^{\pi}}{1+n^2}\,[-\cos n\,\pi] - \frac{ne^{-\pi}}{1+n^2}\,[-\cos(-n\pi)\right\}$$

$$= \frac{1}{\pi}\left[-\frac{ne^{\pi}}{1+n^2}(-1)^n + \frac{ne^{-\pi}}{1+n^2}(-1)^n\right]$$

$$= \frac{n(-1)^n}{\pi(1+n^2)}\,[-e^{\pi} + e^{-\pi}]$$

Now Fourier Series

$$f(x) = \frac{1}{2}\,a_0 + \sum_{1}^{\infty}(\,a_n \cos nx + b_n \sin nx) \qquad \text{becomes}$$

$$f(x) = \frac{1}{2\pi}(e^{\pi} - e^{-\pi}) + \sum_{1}^{\infty}\left[\frac{(-1)^n}{\pi(1+n^2)}(e^{\pi} - e^{-\pi})\cos nx + \frac{n(-1)^n}{\pi(1+n)^2}(-e^{\pi} + e^{-\pi})\sin nx\right]$$

$$= \frac{1}{2\pi}(e^{\pi} - e^{-\pi}) + \frac{e^{\pi} - e^{-\pi}}{\pi}\sum_{1}^{\infty}\frac{(-1)^n}{1+n^2}\,(\cos nx - n\sin nx)$$

$$= \frac{1}{\pi}(e^{\pi} - e^{-\pi})\left\{\frac{1}{2} + \sum_{1}^{\infty}\frac{(-1)^n}{1+n^2}\,(\cos nx - n\sin nx)\right\}$$

Example 4: Find the Fourier Series for

$$f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 0, & x = 0 \\ 1, & 0 < x < \pi \end{cases}$$

Solution: $a_0 = \frac{1}{\pi}\int_{-\pi}^{0}(-1)dx + 0 + \frac{1}{\pi}\int_{0}^{\pi}(1)\,dx$

$$a_0 = \frac{1}{\pi}\left(\left|-x\right|_{-\pi}^{0} + \left|x\right|_{0}^{\pi}\right)$$

$$= \frac{1}{\pi}(-0-\pi) + \frac{1}{\pi}(\pi-0) = -1+1 = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} -1.\cos n\,x\,dx + 0 + \frac{1}{\pi}\int_{0}^{\pi} 1.\cos nx\,dx$$

$$= -\frac{1}{\pi}\left|\frac{\sin nx}{n}\right|_{-\pi}^{0} + \frac{1}{\pi}\left|\frac{\sin nx}{n}\right|_{0}^{\pi}$$

$$= -\frac{1}{\pi}(0) + \frac{1}{\pi}(0) = 0$$

Now $b_n = \frac{1}{\pi}\int_{-\pi}^{0}(-1)\sin nx\,dx + 0 + \frac{1}{\pi}\int_{0}^{\pi} 1\sin n\,x\,dx$

$$= \frac{1}{\pi}\left|\frac{\cos nx}{n}\right|_{-\pi}^{0} + \frac{1}{\pi}\left|-\frac{\cos nx}{n}\right|_{0}^{\pi}$$

$$= \frac{1}{n\pi}[1-\cos(-n\pi)] + \frac{1}{n\pi}(-\cos n\pi + \cos 0)$$

$\therefore \quad \text{Cos}(-n\pi) = \cos n\pi$

$$= \frac{1}{n\pi}(1-(-1)^n) + \frac{1}{n\pi}[-(-1)^n + 1]$$

$$b_n = \frac{2}{n\pi}(1-(-1)^n) = \frac{2}{n\pi}[1+(-1)^{n+1}]$$

So Fourier series becomes

$$f(x) = \frac{1}{2}(0) + \sum_{1}^{\infty}\left[0\cos nx + \frac{2}{n\pi}\{1+(-1)^{n+1}\}\sin nx\right]$$

$$= 0 + \sum_{1}^{\infty}\frac{1}{n\pi}[2+2(-1)^{n+1}]\sin nx$$

$$= \frac{4}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\ldots\right)$$

Example 5:

Find the Fourier series for

$$f(x) = \sin x \quad , \qquad -\pi \le x \le \pi$$

Solution:

We know that sin x is an odd function and its coefficient for Fourier series will be $a_0 = a_n = 0$ and

b_n will be evaluated

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin n\,x\,dx \;=\; \frac{2}{\pi}\int_{0}^{\pi} \sin x \sin nx\,dx$$

$$= -\frac{2}{\pi}\int_{0}^{\pi} -\sin x \sin nx\,dx = -\frac{1}{\pi}\int_{0}^{\pi} -2\sin x \sin nx\,dx$$

$$= -\frac{1}{\pi}\int_{0}^{\pi} \left[\cos(1+n)x - \cos(1-n)x\right] dx$$

$$= -\frac{1}{\pi}\left[\frac{\sin((1+n))x}{(1+n)} - \frac{\sin(1-n)x}{(1-n)}\right]_0^{\pi} = 0$$

For all values of n except n=1

$\therefore \sin 0 = 0$ and $\sin \pi = 0$

For n = 1

$$b_1 = \frac{2}{\pi}\int_{0}^{\pi} \sin^2 x\,dx = \frac{2}{2\pi}\int_{0}^{\pi} (1-\cos 2x)\,dx$$

$$= \frac{1}{\pi}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi} = \frac{1}{\pi}(\pi - 0) = 1$$

So this series becomes

$$f(x) = \frac{1}{2}(0) + \sum_{1}^{\infty}\left(0\cos nx + 1\sin nx\right) = \sum_{1}^{\infty} \text{Sin } nx$$

$$= \text{Sin } x + \text{Sin } 2x + \text{Sin } 3x + \ldots\ldots\ldots\ldots\ldots\ldots$$

Exercise 11

Q.1: Expand the following functions in Fourier Series.

(i) $f(x) = x, \qquad -\pi \le x \le \pi$

(ii) $f(x) = x.\sin x \qquad -\pi \le x \le \pi$

(iii) $f(x) = e^{2ax}, \qquad -\pi \le x \le \pi$

Q.2: Expand the following

(i) $f(x) = \begin{cases} 1, & 0 \le x \le \pi \\ 2, & \pi \le x \le 2\pi \end{cases}$

(ii) $f(x) = \begin{cases} \frac{-\pi}{4}, & -\pi < x < 0 \\ \frac{\pi}{4}, & 0 < x \le \pi \end{cases}$

(iii) $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ 1, & 0 < x < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < x < \pi \end{cases}$ Period 2π

Q.3: Expand the following functions in the Fourier Series.

$f(x) = |x|$, $-\pi \le x \le \pi$

Also deduce that $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots\ldots\ldots$

Q.4: Expand $f(x) = x + x^2$ in Fourier Series in $-\pi \le x \le \pi$

Also deduce that $\frac{\pi}{4} + \frac{\pi^2}{6} = \sum_{1}^{\infty} \frac{1}{n^2}$

Q.5: Obtain Fourier Series for $f(x) = \cos \alpha x$, $-\pi < x \le \pi$.

Also deduce that $\cot \alpha\pi = \frac{1}{\pi}\left(\frac{1}{\alpha} - \sum_{1}^{\infty} \frac{2\alpha}{n^2 - \alpha^2}\right)$, Here α is not an integer

Q.6: Obtain Fourier Series for $f(x) = \frac{\pi}{4}$, $0 < x < \pi$

Also prove that: $\frac{\pi}{4} = \frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5}$

Answers 11

Q.1: (i) $2\left\{\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \ldots\ldots\ldots\right\}$

(ii) $1 + 2\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2 - 1} \cos nx$

(iii) $\frac{e^{2ax}+e^{-2ax}}{\pi}\left\{\frac{1}{4a}+\sum_{1}^{\infty}\frac{(-1)n}{4a^2+n^2}(2a\cos nx - n\sin nx)\right\}$

Q.2: (i) $\frac{3}{2}-\frac{2}{\pi}\left[\frac{\sin x}{1}+\frac{\sin 3x}{3}+\ldots\ldots\right]$

(ii) $\frac{\sin x}{1}+\frac{\sin 3x}{3}+\frac{\sin 5x}{3}+\ldots\ldots.$

(iii) $\frac{1}{4}+\frac{1}{\pi}\left[\frac{\cos x}{1}+\frac{\sin x}{1}-\frac{\cos 3x}{3}+\frac{\sin 3x}{3}+\ldots\ldots\right]$

$+\frac{2}{\pi}\left[\frac{\sin 2x}{2}+\frac{\sin 6x}{6}+\ldots\ldots\right]$

Q.3: $f(x)=\frac{1}{2}-\frac{4}{\pi^2}\left(\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}-\,-\,-\,-\right)$

Deduction for $x = 0, f(x) = 0 \Rightarrow \frac{\pi^2}{8}=1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\ldots..$

Q.4: $f(x)=x+x^2=\frac{\pi^2}{3}+\sum_{1}^{\infty}\frac{4}{n^2}(-1)^n\cos nx+\frac{2}{n}(-1)^{n+1}\sin nx$

Deduction: Put $x = \pi$

$\frac{\pi}{4}+\frac{\pi^2}{6}=\sum_{1}^{\infty}\frac{1}{n^2}$

Q.5: $f(x)=\cot\alpha x=\frac{\sin\alpha x}{\alpha\pi}+\sum_{1}^{\infty}\frac{2(-1)^n\alpha\sin\alpha\pi\cos nx}{\pi(\alpha^2-n^2)}$

Deduction: Put $x = \pi$

$\cot\alpha\pi=\frac{1}{\pi}\left(\frac{1}{\alpha}-\sum_{1}^{\infty}\frac{2\alpha}{n^2-\alpha^2}\right)$

Q.6: $f(x)=\frac{\pi}{4}=\frac{\pi}{8}+\sum_{1}^{\infty}\frac{1}{2n}\sin nx$

Deduction: $\frac{\pi}{2}=\frac{\pi}{4}+\sum_{1}^{\infty}\frac{1}{n}\sin nx$

$\frac{\pi}{4}=\sin x+\frac{\sin 3x}{3}+\frac{\sin 5x}{5}$