$$= \frac{\dfrac{\sin\theta/2}{\cos\theta/2} + \dfrac{\cos\theta/2}{\sin\theta/2}}{\dfrac{\cos\theta/2}{\sin\theta/2} - \dfrac{\sin\theta/2}{\cos\theta/2}}$$

$$= \frac{\dfrac{\sin^2\theta/2 + \cos^2\theta/2}{\cancel{\sin\theta/2\cos\theta/2}}}{\dfrac{\cos^2\theta/2 - \sin^2\theta/2}{\cancel{\sin\theta/2\cos\theta/2}}} \qquad \because \cos\theta = \cos^2\theta/2 - \sin^2\theta/2$$

$$= \frac{1}{\cos\theta}$$

$$= \sec\theta = RHS \text{ (Proved)}$$

Q.13 $\dfrac{\sin 3\theta}{\cos\theta} + \dfrac{\cos 3\theta}{\sin\theta} = 2\cot 2\theta$

$$LHS = \frac{\sin 3\theta}{\cos\theta} + \frac{\cos 3\theta}{\sin\theta}$$

$$= \frac{\cos 3\theta\cos\theta + \sin 3\theta\sin\theta}{\sin\theta\cos\theta}$$

$$= 2\cdot\frac{\cos(3\theta-\theta)}{2\sin\theta\cos\theta}$$

$$= 2\,\frac{\cos 2\theta}{\sin 2\theta}$$

$$= 2\cot 2\theta = RHS \text{ (Proved)}$$

Q.14 Express $\sin^4\theta$ in unity power.

Sol:- $\sin^4\theta = (\sin^2\theta)^2$

$$= \left[\frac{1-\cos 2\theta}{2}\right]^2 \qquad \because \cos 2\theta = 1 - 2\sin^2\theta,\ 2\sin^2\theta = 1-\cos 2\theta$$

$$= \frac{1+\cos^2 2\theta - 2\cos 2\theta}{4} \qquad \because \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$= \frac{1+\left(\dfrac{1+\cos 4\theta}{2}\right) - 2\cos 2\theta}{4}$$

$$= \frac{2+1+\cos 4\theta - 4\cos 2\theta}{8}$$

$$\sin^4\theta = \frac{1}{8}\left[3 - 4\cos 2\theta + \cos 4\theta\right] \text{ Ans}$$

Sum, Difference to Product Formulae

We know that

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ —— (1)

$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$ —— (2)

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ —— (3)

$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ —— (4)

Adding both Sines formula

$\boxed{\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin\alpha\cos\beta}$

Subtracting (2) from (1)

$\boxed{\sin(\alpha+\beta) - \sin(\alpha-\beta) = 2\cos\alpha\cdot\sin\beta}$

Adding both Cosines formula

$\boxed{\cos(\alpha+\beta) + \cos(\alpha-\beta) = 2\cos\alpha\cos\beta}$

Subtracting (4) from (3)

$\boxed{\cos(\alpha+\beta) - \cos(\alpha-\beta) = -2\sin\alpha\cdot\sin\beta}$

let $P = \alpha+\beta$ and $Q = \alpha-\beta$

$\alpha = \dfrac{P+Q}{2}$ and $\beta = \dfrac{P-Q}{2}$

$$\boxed{\begin{aligned}
\sin P + \sin Q &= 2\sin\left(\tfrac{P+Q}{2}\right)\cos\left(\tfrac{P-Q}{2}\right)\\
\sin P - \sin Q &= 2\cos\left(\tfrac{P+Q}{2}\right)\sin\left(\tfrac{P-Q}{2}\right)\\
\cos P + \cos Q &= 2\cos\left(\tfrac{P+Q}{2}\right)\cos\left(\tfrac{P-Q}{2}\right)\\
\cos P - \cos Q &= -2\sin\left(\tfrac{P+Q}{2}\right)\sin\left(\tfrac{P-Q}{2}\right)
\end{aligned}}$$

Now

S+S = 2SC

S−S = 2CS

C+C = 2CC

C−C = −2SS

* From Sum, difference to product Divide Sum and difference of angle to 2 but not conversely.

EXERCISE: 10.4

Q.1 Express as Sum or difference:

(i) $2\sin 3\theta \cos\theta$

$= \sin(3\theta+\theta) + \sin(3\theta-\theta)$

$= \sin 4\theta + \sin 2\theta$

(ii) $2\cos 5\theta \sin 3\theta$

$= \sin(5\theta+3\theta) - \sin(5\theta-3\theta)$

$= \sin 8\theta - \sin 2\theta$

(iii) $\sin 5\theta \cos 2\theta$

$= \frac{1}{2}[2\sin 5\theta \cos 2\theta]$

$= \frac{1}{2}[\sin(5\theta+2\theta) + \sin(5\theta-2\theta)]$

$= \frac{1}{2}[\sin 7\theta + \sin 3\theta]$

(iv) $2\sin 7\theta \sin 2\theta$

$= -[-2\sin 7\theta \sin 2\theta]$

$= -[\cos(7\theta+2\theta) - \cos(7\theta-2\theta)]$

$= -\cos 9\theta + \cos 5\theta$

$= \cos 5\theta - \cos 9\theta.$

(v) $\cos(x+y)\cdot\sin(x-y)$

$= \frac{1}{2}[2\cos(x+y)\cdot\sin(x-y)]$

$= \frac{1}{2}[\sin(x+y+x-y) - \sin(x+y-x+y)]$

$= \frac{1}{2}[\sin 2x - \sin 2y].$

(vi) $\cos(2x+30^\circ)\cdot\cos(2x-30^\circ)$

$= \frac{1}{2}[2\cos(2x+30^\circ)\cos(2x-30^\circ)]$

$= \frac{1}{2}[\cos(2x+30^\circ+2x-30^\circ) + \cos(2x+30^\circ-2x+30^\circ)]$

$= \frac{1}{2}[\cos 4x + \cos 60^\circ]$

(vii) $\sin 46^\circ \sin 12^\circ$

$= -\frac{1}{2}[-2\sin 46^\circ \sin 12^\circ]$

$= -\frac{1}{2}[\cos(46^\circ+12^\circ) - \cos(46^\circ-12^\circ)]$

$= \frac{1}{2}[\cos 34^\circ - \cos 58^\circ]$

Q.(viii) $\sin(x+45^\circ)\sin(x-45^\circ)$

$= -\frac{1}{2}[-2\sin(x+45^\circ)\sin(x-45^\circ)]$

$= -\frac{1}{2}[\cos(x+45^\circ+x-45^\circ) - \cos(x+45^\circ-x+45^\circ)]$

$= -\frac{1}{2}[\cos 2x - \cos 90^\circ]$

$= \frac{1}{2}[\cos 90^\circ - \cos 2x].$

Q.2 Express as product:

(i) $\sin 5\theta + \sin 3\theta$

$= 2\sin\left(\frac{5\theta+3\theta}{2}\right)\cos\left(\frac{5\theta-3\theta}{2}\right)$

$= 2\sin 4\theta\cdot\cos\theta.$

(ii) $\sin 8\theta - \sin 4\theta$

$= 2\cos\left(\frac{8\theta+4\theta}{2}\right)\sin\left(\frac{8\theta-4\theta}{2}\right)$

$= 2\cos 6\theta \sin 2\theta.$

(iii) $\cos 6\theta + \cos 3\theta$

$= 2\cos\left(\frac{6\theta+3\theta}{2}\right)\cos\left(\frac{6\theta-3\theta}{2}\right)$

$= 2\cos\left(\frac{9}{2}\theta\right)\cos\left(\frac{3}{2}\theta\right).$

(iv) $\cos 7\theta - \cos\theta$

$= -2\sin\left(\frac{7\theta+\theta}{2}\right)\sin\left(\frac{7\theta-\theta}{2}\right)$

$= -2\sin 4\theta\cdot\sin 3\theta$

(v) $\cos 48^\circ + \cos 12^\circ$

$= 2\cos\left(\frac{48^\circ+12^\circ}{2}\right)\cos\left(\frac{48^\circ-12^\circ}{2}\right)$

$= 2\cos 30^\circ \cos(18^\circ)$

(vi) $\sin(x+30^\circ) + \sin(x-30^\circ)$

$= 2\sin\left(\frac{x+30^\circ+x-30^\circ}{2}\right)\cos\left(\frac{\cancel{x}+30^\circ-\cancel{x}+30^\circ}{2}\right)$

$= 2\sin\left(\frac{2x}{2}\right)\cos\left(\frac{60^\circ}{2}\right)$

$= 2\sin x\cos(30^\circ)$

Q.3 Prove that:

(i) $\dfrac{\sin 3x - \sin x}{\cos x - \cos 3x} = \cot 2x$

$$LHS = \frac{\sin 3x - \sin x}{\cos x - \cos 3x}$$

$$= \frac{2\cos\left(\frac{3x+x}{2}\right)\sin\left(\frac{3x-x}{2}\right)}{-2\sin\left(\frac{x+3x}{2}\right)\sin\left(\frac{x-3x}{2}\right)}$$

$$= -\frac{\cos 2x \sin x}{\sin 2x \sin(-x)}$$

$\because \sin(-\theta) = -\sin\theta$

$$= -\frac{\cos 2x \cdot \cancel{\sin x}}{-\sin 2x \cdot \cancel{\sin x}}$$

$= \cot 2x = RHS$ (Proved)

(ii) $\dfrac{\sin 8x + \sin 2x}{\cos 8x + \cos 2x} = \tan 5x$

$$LHS = \frac{\sin 8x + \sin 2x}{\cos 8x + \cos 2x}$$

$$= \frac{\cancel{2}\sin\left(\frac{8x+2x}{2}\right)\cancel{\cos\left(\frac{8x-2x}{2}\right)}}{\cancel{2}\cos\left(\frac{8x+2x}{2}\right)\cancel{\cos\left(\frac{8x-2x}{2}\right)}}$$

$= \dfrac{\sin 5x}{\cos 5x} = \tan 5x = RHS$ (Proved)

(iii) $\dfrac{\sin\alpha - \sin\beta}{\sin\alpha + \sin\beta} = \tan\left(\frac{\alpha-\beta}{2}\right)\cot\left(\frac{\alpha+\beta}{2}\right)$

$$LHS = \frac{\sin\alpha - \sin\beta}{\sin\alpha + \sin\beta}$$

$$= \frac{\cancel{2}\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)}{\cancel{2}\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)}$$

$$= \cot\left(\frac{\alpha+\beta}{2}\right)\cdot\tan\left(\frac{\alpha-\beta}{2}\right)$$

$= RHS$. (Proved)

Q.4 Prove that

(i) $\cos 20° + \cos 100° + \cos 140° = 0$

$LHS = \cos 140° + \cos 100° + \cos 20°$

$= 2\cos\left(\frac{140°+100°}{2}\right)\cos\left(\frac{140°-100°}{2}\right) + \cos 20°$

$= 2\cos(120°)\cos 20° + \cos 20°$

$= (2\cos 120° + 1)\cos 20°$

$= \left[2\left(\frac{-1}{2}\right)+1\right]\cos 20°$

$= [-1+1]\cos 20°$

$= (0).\cos 20°$

$= 0 = RHS$ (Proved)

(ii) $\sin\left(\frac{\pi}{4}-\theta\right)\cdot\sin\left(\frac{\pi}{4}+\theta\right) = \frac{1}{2}\cos 2\theta$.

$LHS = \sin\left(\frac{\pi}{4}+\theta\right)\cdot\sin\left(\frac{\pi}{4}-\theta\right)$

$= -\frac{1}{2}\left[-2\sin\left(\frac{\pi}{4}+\theta\right)\sin\left(\frac{\pi}{4}-\theta\right)\right]$

$= -\frac{1}{2}\left[\cos\left(\frac{\pi}{4}+\cancel{\theta}+\frac{\pi}{4}-\cancel{\theta}\right) - \cos\left(\cancel{\frac{\pi}{4}}+\theta-\cancel{\frac{\pi}{4}}+\theta\right)\right]$

$= -\frac{1}{2}\left[\cos\left(\frac{\pi}{2}\right) - \cos 2\theta\right]$

$= \frac{-1}{2}\left[0 - \cos 2\theta\right]$ $\quad \because \cos\frac{\pi}{2} = 0$

$= \frac{1}{2}\cos 2\theta = RHS$. (Proved)

(iii) $\dfrac{\sin 7\theta + \sin 5\theta + \sin 3\theta + \sin\theta}{\cos 7\theta + \cos 5\theta + \cos 3\theta + \cos\theta} = \tan 4\theta$

$$LHS = \frac{(\sin 7\theta + \sin\theta) + (\sin 5\theta + \sin 3\theta)}{(\cos 7\theta + \cos\theta) + (\cos 5\theta + \cos 3\theta)}$$

$$= \frac{2\sin\left(\frac{7\theta+\theta}{2}\right)\cos\left(\frac{7\theta-\theta}{2}\right) + 2\sin\left(\frac{5\theta+3\theta}{2}\right)\cos\left(\frac{5\theta-3\theta}{2}\right)}{2\cos\left(\frac{7\theta+\theta}{2}\right)\cos\left(\frac{7\theta-\theta}{2}\right) + 2\cos\left(\frac{5\theta+3\theta}{2}\right)\cos\left(\frac{5\theta-3\theta}{2}\right)}$$

$$= \frac{2\sin 4\theta\cos 3\theta + 2\sin 4\theta\cos\theta}{2\cos 4\theta\cos 3\theta + 2\cos 4\theta\cos\theta}$$

$$= \frac{\cancel{2}\sin 4\theta\cancel{(\cos 3\theta + \cos\theta)}}{\cancel{2}\cos 4\theta\cancel{(\cos 3\theta + \cos\theta)}} = \tan 4\theta$$

$= RHS$ (Proved)

Q.5/ Prove that:

(i) $\cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ = \frac{1}{16}$

Consider $\cos 60^\circ \cos 40^\circ \cos 20^\circ \cos 80^\circ$

$= \left(\frac{1}{2}\right) \cos 40^\circ \cos 20^\circ \cdot \cos 80^\circ$

$= \frac{1}{4}\left[2\cos 40^\circ \cos 20^\circ\right] \cos 80^\circ$

$= \frac{1}{4}\left[\cos(40^\circ+20^\circ) + \cos(40^\circ-20^\circ)\right] \cos 80^\circ$

$= \frac{1}{4}\left[\cos 60^\circ + \cos 20^\circ\right] \cos 80^\circ$

$= \frac{1}{4}\left[\frac{1}{2} + \cos 20^\circ\right] \cos 80^\circ$

$= \frac{1}{4}\left[\frac{1+2\cos 20^\circ}{2}\right] \cos 80^\circ$

$= \frac{1}{8}\left[\cos 80^\circ + 2\cos 80^\circ \cos 20^\circ\right]$

$= \frac{1}{8}\left[\cos 80 + \cos(80^\circ+20^\circ) + \cos(80^\circ-20^\circ)\right]$

$= \frac{1}{8}\left[\cos 80^\circ + \cos 100^\circ + \cos 60^\circ\right]$

$= \frac{1}{8}\left[\cos 80 + \cos(180^\circ-80^\circ) + \frac{1}{2}\right]$

$= \frac{1}{8}\left[\cancel{\cos 80} - \cancel{\cos 80^\circ} + \frac{1}{2}\right]$

$= \frac{1}{16}$ (Proved) $\quad \cos(180-\theta) = -\cos\theta$

(ii) $\sin\frac{\pi}{9} \sin\frac{2\pi}{9} \sin\frac{\pi}{3} \sin\frac{4\pi}{9} = \frac{3}{16}$

$\because \pi = 180^\circ \Rightarrow \frac{\pi}{9} = 20^\circ$

Consider $\sin\frac{\pi}{9} \cdot \sin\frac{2\pi}{9} \cdot \sin\frac{\pi}{3} \sin\frac{4\pi}{9}$

$= \sin 20^\circ \cdot \sin 40^\circ \cdot \sin 60^\circ \cdot \sin 80^\circ$

$= \sin 60^\circ \left[\sin 40^\circ \sin 20^\circ\right] \sin 80^\circ$

$= \frac{\sqrt{3}}{2}\left[\sin 40^\circ \sin 20^\circ\right] \sin 80^\circ$

$= -\frac{\sqrt{3}}{4}\left[-2\sin 40^\circ \sin 20^\circ\right] \sin 80^\circ$

$= -\frac{\sqrt{3}}{4}\left[\cos(40^\circ+20^\circ) - \cos(40^\circ-20^\circ)\right] \sin 80^\circ$

$= -\frac{\sqrt{3}}{4}\left[\cos 60^\circ - \cos 20^\circ\right] \sin 80^\circ$

$= -\frac{\sqrt{3}}{4}\left[\frac{1}{2} - \cos 20^\circ\right] \sin 80^\circ$

$= -\frac{\sqrt{3}}{4}\left[\frac{1-2\cos 20^\circ}{2}\right] \sin 80^\circ$

$= -\frac{\sqrt{3}}{8}\left[\sin 80^\circ - 2\sin 80^\circ \cos 20^\circ\right]$

$= -\frac{\sqrt{3}}{8}\left[\sin 80^\circ - (\sin(80^\circ+20^\circ) + \sin(80-20^\circ))\right]$

$= -\frac{\sqrt{3}}{8}\left[\sin 80^\circ - \sin 100^\circ - \sin 60^\circ\right]$

$= -\frac{\sqrt{3}}{8}\left[\sin 80^\circ - \sin(180^\circ-80^\circ) - \frac{\sqrt{3}}{2}\right]$

$= -\frac{\sqrt{3}}{8}\left[\cancel{\sin 80^\circ} - \cancel{\sin 80^\circ} - \frac{\sqrt{3}}{2}\right]$

$\because \sin(180^\circ-\theta) = \sin\theta$

$= -\frac{\sqrt{3}}{8} \times -\frac{\sqrt{3}}{2} = \frac{3}{16}$ (Proved)

(iii) $\sin 10^\circ \sin 30^\circ \sin 50^\circ \sin 70^\circ = \frac{1}{16}$

Consider $\sin 10^\circ \sin 30^\circ \sin 50^\circ \sin 70^\circ$

$= \sin 30^\circ \left[\sin 50^\circ \sin 10^\circ\right] \sin 70^\circ$

$= \frac{1}{2}\left[\sin 50^\circ \sin 10^\circ\right] \sin 70^\circ$

$= -\frac{1}{4}\left[-2\sin 50^\circ \sin 10^\circ\right] \sin 70^\circ$

$= -\frac{1}{4}\left[\cos(50^\circ+10^\circ) - \cos(50^\circ-10^\circ)\right] \sin 70^\circ$

$= -\frac{1}{4}\left[\cos 60^\circ - \cos 40^\circ\right] \sin 70^\circ$

$= -\frac{1}{4}\left[\frac{1}{2} - \cos 40^\circ\right] \sin 70^\circ$

$= -\frac{1}{4}\left[\frac{1-2\cos 40^\circ}{2}\right] \sin 70^\circ$

$= -\frac{1}{8}\left[\sin 70^\circ - 2\sin 70^\circ \cos 40^\circ\right]$

$= -\frac{1}{8}\left[\sin 70^\circ - (\sin(70^\circ+40^\circ) + \sin(70^\circ-40^\circ))\right]$

$= -\frac{1}{8}\left[\sin 70^\circ - \sin 110^\circ - \sin 30^\circ\right]$

$= -\frac{1}{8}\left[\sin 70^\circ - \sin(180^\circ-70^\circ) - \frac{1}{2}\right]$

$\because \sin(180^\circ-\theta) = \sin\theta$

$= -\frac{1}{8}\left[\cancel{\sin 70} - \cancel{\sin 70^\circ} - \frac{1}{2}\right]$

$= \frac{1}{16}$ (Proved)

THE End

Tahir Mahmood
M.Sc. (Math)
0345-6510779