EXERCISE: 10.2

Q.1 Prove that:

(i) $\sin(180^\circ+\theta) = -\sin\theta$

LHS $= \sin(180^\circ+\theta)$

$= \sin 180^\circ \cos\theta + \cos 180^\circ \sin\theta$

$= (0)\cos\theta + (-1)\sin\theta$

$= -\sin\theta = \text{RHS}$ (Proved)

(ii) $\cos(180^\circ+\theta) = -\cos\theta$

LHS $= \cos(180^\circ+\theta)$

$= \cos 180^\circ \cos\theta - \sin 180^\circ \sin\theta$

$= (-1)\cos\theta - (0)\sin\theta$

$= -\cos\theta = \text{RHS}$ (Proved)

(iii) $\tan(270^\circ-\theta) = +\cot\theta$

LHS $= \tan(270^\circ-\theta)$

$= \dfrac{\sin(270^\circ-\theta)}{\cos(270^\circ-\theta)}$

$= \dfrac{\sin 270^\circ \cos\theta - \cos 270^\circ \sin\theta}{\cos 270^\circ \cos\theta + \sin 270^\circ \sin\theta}$

$= \dfrac{(-1)\cos\theta - (0)\sin\theta}{(0)\cos\theta + (-1)\sin\theta} = \dfrac{-\cos\theta}{-\sin\theta}$

$= \cot\theta = \text{RHS}$ (Proved)

(iv) $\cos(\theta-180^\circ) = -\cos\theta$

LHS $= \cos(\theta-180^\circ)$

$= \cos\theta \cos 180^\circ + \sin\theta \sin 180^\circ$

$= \cos\theta(-1) + \sin\theta(0)$

$= -\cos\theta = \text{RHS}$ (Proved)

(v) $\cos(270^\circ+\theta) = \sin\theta$

LHS $= \cos(270^\circ+\theta)$

$= \cos 270^\circ \cos\theta - \sin 270^\circ \sin\theta$

$= (0)\cos\theta - (-1)\sin\theta$

$= \sin\theta = \text{RHS}$ (Proved)

(vi) $\sin(\theta+270^\circ) = -\cos\theta$

LHS $= \sin(\theta+270^\circ)$

$= \sin\theta \cos 270^\circ + \cos\theta \sin 270^\circ$

$= \sin\theta(0) + \cos\theta(-1)$

$= -\cos\theta = \text{RHS}$ (Proved)

(vii) $\tan(180^\circ+\theta) = \tan\theta$

LHS $= \tan(180^\circ+\theta)$

$= \dfrac{\tan 180^\circ + \tan\theta}{1 - \tan 180^\circ \tan\theta}$

$= \dfrac{0+\tan\theta}{1-(0)\tan\theta} = \dfrac{0+\tan\theta}{1-0}$

$= \tan\theta = \text{RHS}$ (Proved)

(viii) $\cos(360^\circ-\theta) = \cos\theta$

LHS $= \cos(360^\circ-\theta)$

$= \cos 360^\circ \cos\theta + \sin 360^\circ \sin\theta$

$= (1)\cos\theta + (0)\sin\theta$

$= \cos\theta = \text{RHS}$ (Proved)

Q.2 Find the value of:

(i) $\sin 15^\circ = \sin(60^\circ-45^\circ)$

$= \sin 60^\circ \cos 45^\circ - \cos 60^\circ \sin 45^\circ$

$= \dfrac{\sqrt{3}}{2}\cdot\dfrac{1}{\sqrt{2}} - \dfrac{1}{2}\cdot\dfrac{1}{\sqrt{2}}$

$= \dfrac{\sqrt{3}-1}{2\sqrt{2}}$ Ans.

(ii) $\cos(15^\circ) = \cos(60^\circ-45^\circ)$

$= \cos 60^\circ \cos 45^\circ + \sin 60^\circ \sin 45^\circ$

$= \dfrac{1}{2}\cdot\dfrac{1}{\sqrt{2}} + \dfrac{\sqrt{3}}{2}\cdot\dfrac{1}{\sqrt{2}}$

$= \dfrac{1+\sqrt{3}}{2\sqrt{2}}$ Ans.

(iii) $\tan 15^\circ = \tan(60^\circ-45^\circ)$

$= \dfrac{\tan 60^\circ - \tan 45^\circ}{1+\tan 60^\circ \tan 45^\circ} = \dfrac{\sqrt{3}-1}{1+1\cdot\sqrt{3}}$

$= \dfrac{\sqrt{3}-1}{\sqrt{3}+1}$ Ans.

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iv) $\sec 15^\circ = \dfrac{1}{\cos 15^\circ}$

$= \dfrac{2\sqrt{2}}{\sqrt{3}+1}$ Ans.

(v) $\sin 105^\circ = \sin(60^\circ+45^\circ)$

$= \sin 60^\circ \cos 45^\circ + \cos 60^\circ \sin 45^\circ$

$= \sqrt{3}/2 \cdot 1/\sqrt{2} + \frac{1}{2} \cdot 1/\sqrt{2}$

$= \dfrac{\sqrt{3}+1}{2\sqrt{2}}$ Ans.

(vi) $\cos 105^\circ = \cos(60^\circ+45^\circ)$

$= \cos 60^\circ \cos 45^\circ - \sin 60^\circ \sin 45^\circ$

$= \frac{1}{2} \cdot \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}}$

$= \dfrac{1-\sqrt{3}}{2\sqrt{2}}$ Ans.

(vii) $\tan 105^\circ = \tan(60^\circ+45^\circ)$

$= \dfrac{\tan 60^\circ + \tan 45^\circ}{1-\tan 60^\circ \tan 45^\circ}$

$= \dfrac{\sqrt{3}+1}{1-1\cdot\sqrt{3}} = \dfrac{\sqrt{3}+1}{1-\sqrt{3}}$ Ans.

(viii) $\sec 105^\circ = 1/\cos 105^\circ$

$= \dfrac{2\sqrt{2}}{1-\sqrt{3}}$ Ans.

Q.3 Prove that:

(i) $\sin(45^\circ+\alpha) = \frac{1}{\sqrt{2}}(\sin\alpha+\cos\alpha)$

LHS $= \sin(45^\circ+\alpha)$

$= \sin 45^\circ \cos\alpha + \cos 45^\circ \sin\alpha$

$= \frac{1}{\sqrt{2}}\cos\alpha + \frac{1}{\sqrt{2}}\sin\alpha$

$= \frac{1}{\sqrt{2}}(\sin\alpha + \cos\alpha) =$ RHS (Proved)

(ii) $\cos(45^\circ+\alpha) = \frac{1}{\sqrt{2}}(\cos\alpha - \sin\alpha)$

LHS $= \cos(\alpha+45^\circ)$

$= \cos\alpha \cos 45^\circ - \sin\alpha \sin 45^\circ$

$= \cos\alpha \cdot \frac{1}{\sqrt{2}} - \sin\alpha \cdot \frac{1}{\sqrt{2}}$

$= \frac{1}{\sqrt{2}}(\cos\alpha - \sin\alpha) =$ RHS. (Proved)

Q.4 Prove that

(i) $\tan(45^\circ+A)\cdot\tan(45^\circ-A) = 1$

LHS $= \tan(45^\circ+A)\cdot\tan(45^\circ-A)$

$= \left(\dfrac{\tan 45^\circ + \tan A}{1-\tan 45^\circ \tan A}\right)\cdot\left(\dfrac{\tan 45^\circ - \tan A}{1+\tan 45^\circ \cdot \tan A}\right)$

$\because \tan 45^\circ = 1$ so

$= \left(\dfrac{1+\tan A}{1-1\cdot\tan A}\right)\times\left(\dfrac{1-\tan A}{1+1\cdot\tan A}\right)$

$= \dfrac{\cancel{1+\tan A}}{\cancel{1-\tan A}} \times \dfrac{\cancel{1-\tan A}}{\cancel{1+\tan A}}$

$= 1 =$ RHS (Proved)

(ii) $\tan(\frac{\pi}{4}-\theta) + \tan(\frac{3\pi}{4}+\theta) = 0$

LHS $= \tan(\pi/4-\theta) + \tan(3\pi/4+\theta)$

$= \dfrac{\tan\pi/4 - \tan\theta}{1+\tan\pi/4\tan\theta} + \dfrac{\tan 3\pi/4 + \tan\theta}{1-\tan\frac{3\pi}{4}\tan\theta}$

$\because \tan\pi/4 = 1$ and $\tan 3\pi/4 = -1$

$= \dfrac{1-\tan\theta}{1+1\cdot\tan\theta} + \dfrac{-1+\tan\theta}{1-(-1)\tan\theta}$

$= \dfrac{1-\tan\theta}{1+\tan\theta} + \dfrac{-1+\tan\theta}{1+\tan\theta}$

$= \dfrac{1-\tan\theta-1+\tan\theta}{1+\tan\theta} = \dfrac{0}{1+\tan\theta}$

$= 0 =$ RHS (Proved)

(iii) $\sin(\theta+\pi/6) + \cos(\theta+\pi/3) = \cos\theta$

LHS $= \sin(\theta+\pi/6) + \cos(\theta+\pi/3)$

$= \sin\theta\cos\pi/6 + \cos\theta\sin\pi/6 + \cos\theta\cos\pi/3 - \sin\theta\sin\pi/3$

$= \cancel{\sin\theta(\sqrt{3}/2)} + \cos\theta(1/2) + \cos\theta(1/2) - \cancel{\sin\theta(\sqrt{3}/2)}$

$= \cos\theta =$ RHS (Proved)

(iv) $\dfrac{\sin\theta - \cos\theta \cdot \tan\theta/2}{\cos\theta + \sin\theta \cdot \tan\theta/2} = \tan\theta/2$

$$LHS = \frac{\sin\theta - \cos\theta \tan\theta/2}{\cos\theta + \sin\theta \tan\theta/2}$$

$$= \frac{\sin\theta - \cos\theta \cdot \frac{\sin\theta/2}{\cos\theta/2}}{\cos\theta + \sin\theta \cdot \frac{\sin\theta/2}{\cos\theta/2}}$$

$$= \frac{\sin\theta \cos\theta/2 - \cos\theta \sin\theta/2}{\cos\theta \cos\theta/2 + \sin\theta \sin\theta/2}$$

$$= \frac{\sin(\theta - \theta/2)}{\cos(\theta - \theta/2)} = \tan\left(\theta - \frac{\theta}{2}\right)$$

$$= \tan\frac{\theta}{2} = RHS \quad (\text{Proved})$$

(v) $\dfrac{1 - \tan\theta \cdot \tan\phi}{1 + \tan\theta \cdot \tan\phi} = \dfrac{\cos(\theta+\phi)}{\cos(\theta-\phi)}$

$$LHS = \frac{1 - \tan\theta \cdot \tan\phi}{1 + \tan\theta \cdot \tan\phi}$$

$$= \frac{1 - \frac{\sin\theta}{\cos\theta} \cdot \frac{\sin\phi}{\cos\phi}}{1 + \frac{\sin\theta}{\cos\theta} \cdot \frac{\sin\phi}{\cos\phi}}$$

$$= \frac{\cos\theta\cos\phi - \sin\theta\sin\phi}{\cos\theta\cos\phi + \sin\theta\sin\phi}$$

$$= \frac{\cos(\theta+\phi)}{\cos(\theta-\phi)} = RHS = (\text{Proved})$$

Q.5 Show that:

$LHS = \cos(\alpha+\beta) \cdot \cos(\alpha-\beta)$

$= (\cos\alpha\cos\beta - \sin\alpha\sin\beta)(\cos\alpha\cos\beta + \sin\alpha\sin\beta)$

$= \cos^2\alpha\cos^2\beta - \sin^2\alpha\sin^2\beta$

$= \cos^2\alpha(1 - \sin^2\beta) - (1 - \cos^2\alpha)\sin^2\beta$

$= \cos^2\alpha - \cancel{\cos^2\alpha\sin^2\beta} - \sin^2\beta + \cancel{\cos^2\alpha\sin^2\beta}$

$= \cos^2\alpha - \sin^2\beta = RHS \quad (\text{Proved})$

(ii) $\cos(\alpha+\beta)\cos(\alpha-\beta) = \cos^2\alpha - \sin^2\beta$

$= (1 - \sin^2\alpha) - (1 - \cos^2\beta)$

$= 1 - \sin^2\alpha - 1 + \cos^2\beta$

$= \cos^2\beta - \sin^2\alpha \quad (\text{Proved})$

Q.6 Prove that:

$$\frac{\sin(\alpha+\beta) + \sin(\alpha-\beta)}{\cos(\alpha+\beta) + \cos(\alpha-\beta)} = \tan\alpha$$

$$LHS = \frac{\sin(\alpha+\beta) + \sin(\alpha-\beta)}{\cos(\alpha+\beta) + \cos(\alpha-\beta)}$$

$$= \frac{(\sin\alpha\cos\beta + \cancel{\cos\alpha\sin\beta}) + (\sin\alpha\cos\beta - \cancel{\cos\alpha\sin\beta})}{(\cos\alpha\cos\beta - \cancel{\sin\alpha\sin\beta}) + (\cos\alpha\cos\beta + \cancel{\sin\alpha\sin\beta})}$$

$$= \frac{\cancel{2}\sin\alpha\cancel{\cos\beta}}{\cancel{2}\cos\alpha\cancel{\cos\beta}} = \frac{\sin\alpha}{\cos\alpha}$$

$$= \tan\alpha = RHS \quad (\text{Proved})$$

Q.7 Prove that:

(i) $\cot(\alpha+\beta) = \dfrac{\cot\alpha\cot\beta - 1}{\cot\alpha + \cot\beta}$

$$RHS = \frac{\cot\alpha\cot\beta - 1}{\cot\alpha + \cot\beta}$$

$$= \frac{\frac{\cos\alpha}{\sin\alpha} \cdot \frac{\cos\beta}{\sin\beta} - 1}{\frac{\cos\alpha}{\sin\alpha} + \frac{\cos\beta}{\sin\beta}}$$

$$= \frac{\cos\alpha\cos\beta - \sin\alpha\sin\beta}{\cos\alpha\sin\beta + \sin\alpha\cos\beta}$$

$$= \frac{\cos(\alpha+\beta)}{\sin(\alpha+\beta)} = \cot(\alpha+\beta) = LHS \quad (\text{Proved})$$

(ii) $\cot(\alpha-\beta) = \dfrac{\cot\alpha \cdot \cot\beta + 1}{\cot\beta - \cot\alpha}$

$$RHS = \frac{\cot\alpha \cdot \cot\beta + 1}{\cot\beta - \cot\alpha}$$

$$= \frac{\frac{\cos\alpha}{\sin\alpha} \cdot \frac{\cos\beta}{\sin\beta} + 1}{\frac{\cos\beta}{\sin\beta} - \frac{\cos\alpha}{\sin\alpha}}$$

$$= \frac{\cos\alpha\cos\beta + \sin\alpha\sin\beta}{\sin\alpha\cos\beta - \cos\alpha\sin\beta}$$

$$= \frac{\cos(\alpha-\beta)}{\sin(\alpha-\beta)} = \cot(\alpha-\beta) = LHS \quad (\text{Proved})$$

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(iii) $\dfrac{\tan\alpha+\tan\beta}{\tan\alpha-\tan\beta}=\dfrac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)}$

$$LHS=\frac{\tan\alpha+\tan\beta}{\tan\alpha-\tan\beta}$$

$$=\frac{\frac{\sin\alpha}{\cos\alpha}+\frac{\sin\beta}{\cos\beta}}{\frac{\sin\alpha}{\cos\alpha}-\frac{\sin\beta}{\cos\beta}}$$

$$=\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\sin\alpha\cos\beta-\cos\alpha\sin\beta}$$

$$=\frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)}=RHS$$

(Proved)

Q.8 If $\sin\alpha=4/5$ $\quad 0<\alpha<\pi/2$

$\cos\beta=40/41$ $\quad 0<\beta<\pi/2$

$$\cos\alpha=\sqrt{1-\sin^2\alpha}$$
$$=\sqrt{1-16/25}$$
$$=\sqrt{\frac{25-16}{25}}=\sqrt{\frac{9}{25}}$$
$$\cos\alpha=3/5$$

$$\sin\beta=\sqrt{1-\cos^2\beta}$$
$$=\sqrt{1-\frac{1600}{1681}}$$
$$=\sqrt{\frac{1681-1600}{1681}}=\sqrt{\frac{81}{1681}}$$
$$\sin\beta=9/41$$

$$\because \sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$
$$=\frac{4}{5}\times\frac{40}{41}-\frac{3}{5}\times\frac{9}{41}$$
$$=\frac{160}{205}-\frac{27}{205}=\frac{160-27}{205}$$
$$=\frac{133}{205}$$

$$\sin(\alpha-\beta)=\frac{133}{205}$$ (Proved)

Q.9 $\sin\alpha=4/5$ $\quad \pi/2<\alpha<\pi$

$\sin\beta=\frac{12}{13}$ $\quad \pi/2<\beta<\pi$

$$\cos\alpha=\sqrt{1-\sin^2\alpha}$$
$$=\sqrt{1-\frac{16}{25}}$$
$$=\sqrt{\frac{25-16}{25}}$$
$$=\sqrt{9/25}$$
$$\cos\alpha=3/5$$

$$\cos\beta=\sqrt{1-\sin^2\beta}$$
$$=\sqrt{1-\frac{144}{169}}$$
$$=\sqrt{\frac{169-144}{169}}$$
$$=\sqrt{\frac{25}{169}}$$
$$\cos\beta=5/13$$

$\because \cos\alpha, \cos\beta$ is $-$ve in 2nd Quadrent

$\Rightarrow \cos\alpha=-3/5 \wedge \cos\beta=-5/13$

(i) $\sin(\alpha+\beta)=?$

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$
$$=\left(\frac{4}{5}\right)\times\left(\frac{-5}{13}\right)+\left(\frac{-3}{5}\right)\times\left(\frac{12}{13}\right)$$
$$=\frac{-20}{65}-\frac{36}{65}=\frac{-56}{65}$$
$$\Rightarrow \boxed{\sin(\alpha+\beta)=\frac{-56}{65}}$$

(ii) $\cos(\alpha+\beta)=?$

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$
$$=\left(\frac{-3}{5}\right)\times\left(\frac{-5}{13}\right)-\left(\frac{4}{5}\right)\times\left(\frac{12}{13}\right)$$
$$=\frac{+15}{65}-\frac{36}{65}=\frac{-33}{65}$$
$$\boxed{\cos(\alpha+\beta)=\frac{-33}{65}}$$

(iii) $\tan(\alpha+\beta)=\dfrac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}$

$$=\frac{-56/65}{-33/65}=\frac{56}{33}$$
$$\Rightarrow \boxed{\tan(\alpha+\beta)=\frac{56}{33}}$$

(iv) $\sin(\alpha-\beta)=?$

$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$
$$=(4/5)\times\left(\frac{-5}{13}\right)-\left(\frac{-3}{5}\right)\times\left(\frac{12}{13}\right)$$
$$=\frac{-20}{65}+\frac{36}{65}=\frac{16}{65}$$
$$\Rightarrow \boxed{\sin(\alpha-\beta)=16/65}$$

(v) $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$= \left(\frac{-3}{5}\right)\times\left(\frac{-5}{13}\right) + \left(\frac{4}{5}\right)\times\left(\frac{12}{13}\right)$

$= \frac{15}{65} + \frac{48}{65} = \frac{63}{65}$

$\boxed{\cos(\alpha-\beta) = \frac{63}{65}}$

(vi) $\tan(\alpha-\beta) = \frac{\sin(\alpha-\beta)}{\cos(\alpha-\beta)}$

$= \frac{16/65}{63/65} = \frac{16}{63}$

$\boxed{\tan(\alpha-\beta) = 16/63}$

$\because$ $\cos(\alpha+\beta)$ and $\sin(\alpha+\beta)$ are −ve

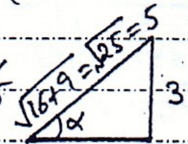
so $\alpha+\beta$ lies in 3rd Quadrent.

$\because$ $\cos(\alpha-\beta)$ and $\sin(\alpha-\beta)$ are +ve

so $\alpha-\beta$ lies in Ist Quadrent.

Q.10 (i) $\tan\alpha = 3/4$ (in III Quad.)

$\sin\alpha = -3/5$ $\cos\alpha = -4/5$



Also $\cos\beta = 5/13$ (in IV Quadreut)

$\sin\beta = \sqrt{1-\cos^2\beta}$

$= \sqrt{1-\frac{25}{169}} = \sqrt{\frac{169-25}{169}}$

$\sin\beta = \sqrt{\frac{144}{169}} = -\frac{12}{13}$ (in IV Quad.)

Now

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$= (-3/5)\times(5/13) + (-4/5)\times(-12/13)$

$= \frac{-15}{65} + \frac{48}{65} = \frac{33}{65}$

$\Rightarrow \boxed{\sin(\alpha+\beta) = \frac{33}{65}}$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

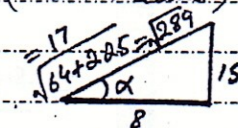
$= \left(\frac{-4}{5}\right)\times\left(\frac{5}{13}\right) - \left(\frac{-3}{5}\right)\times\left(\frac{-12}{13}\right)$

$= \frac{-20}{65} - \frac{36}{65} = \frac{-56}{65}$

$\boxed{\cos(\alpha+\beta) = -56/65}$

(ii) $\tan\alpha = \frac{-15}{8}$ (in IInd Quad.)

$\sin\alpha = \frac{15}{17}$ $\cos\alpha = \frac{-8}{17}$



Also $\sin\beta = \frac{-7}{25}$ (in 3rd Quad.)

$\cos\beta = \sqrt{1-\sin^2\beta} = \sqrt{1-\frac{49}{625}}$

$\cos\beta = \sqrt{\frac{625-49}{625}} = \sqrt{\frac{576}{625}} = \frac{-24}{25}$

Now

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$= \left(\frac{15}{17}\right)\times\left(\frac{-24}{25}\right) + \left(\frac{-8}{17}\right)\times\left(\frac{-7}{25}\right)$

$= \frac{-360}{425} + \frac{56}{425} = \frac{-304}{425}$

$\Rightarrow \boxed{\sin(\alpha+\beta) = -\frac{304}{425}}$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$= \left(\frac{-8}{17}\right)\times\left(\frac{-24}{25}\right) - \left(\frac{15}{17}\right)\times\left(\frac{-7}{25}\right)$

$= \frac{192}{425} + \frac{105}{425} = \frac{297}{425}$

$\boxed{\cos(\alpha+\beta) = \frac{297}{425}}$

Q.11 Prove that:

$\frac{\cos 8^\circ - \sin 8^\circ}{\cos 8^\circ + \sin 8^\circ} = \tan(37^\circ)$

RHS $= \tan 37^\circ$

$= \tan(45^\circ - 8^\circ)$

$= \frac{\tan 45^\circ - \tan 8^\circ}{1 + \tan 45^\circ \tan 8^\circ}$

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$$= \frac{1-\tan 8^\circ}{1+1\cdot\tan 8^\circ}$$

$$= \frac{1-\tan 8^\circ}{1+\tan 8^\circ}$$

$$= \frac{1-\sin 8^\circ/\cos 8^\circ}{1+\sin 8^\circ/\cos 8^\circ}$$

$$= \frac{\cos 8^\circ-\sin 8^\circ}{\cos 8^\circ+\sin 8^\circ} = LHS \text{ (Proved)}$$

Q.12 : α, β, γ are the angles of $\triangle ABC$

So $\alpha+\beta+\gamma = 180^\circ$

$\alpha+\beta = 180^\circ-\gamma$

$\Rightarrow \frac{\alpha}{2}+\frac{\beta}{2} = 90^\circ-\frac{\gamma}{2}$

$\Rightarrow \cot\left(\frac{\alpha}{2}+\frac{\beta}{2}\right) = \cot\left(90-\frac{\gamma}{2}\right)$ $\quad \because \tan\theta = \cot(90-\theta)$

$\Rightarrow \frac{\cot\frac{\alpha}{2}\cot\frac{\beta}{2}-1}{\cot\frac{\alpha}{2}+\cot\frac{\beta}{2}} = \tan\frac{\gamma}{2}$

$\Rightarrow \frac{\cot\frac{\alpha}{2}\cot\frac{\beta}{2}-1}{\cot\frac{\alpha}{2}+\cot\frac{\beta}{2}} = \frac{1}{\cot\frac{\gamma}{2}}$

$\cot\frac{\alpha}{2}\cot\frac{\beta}{2}\cot\frac{\gamma}{2}-\cot\frac{\gamma}{2} = \cot\frac{\alpha}{2}+\cot\frac{\beta}{2}$

$\Rightarrow \cot\frac{\alpha}{2}\cdot\cot\frac{\beta}{2}\cdot\cot\frac{\gamma}{2} = \cot\frac{\alpha}{2}+\cot\frac{\beta}{2}+\cot\frac{\gamma}{2}$ (Proved)

Q.13 : $\alpha+\beta+\gamma = 180^\circ$

$\Rightarrow \alpha+\beta = 180^\circ-\gamma$

$\Rightarrow \cot(\alpha+\beta) = \cot(180^\circ-\gamma)$

$\Rightarrow \frac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta} = -\cot\gamma$

$\Rightarrow \cot\alpha\cot\beta-1 = -\cot\gamma\cot\alpha-\cot\beta\cot\gamma$

$\Rightarrow \cot\alpha\cot\beta+\cot\beta\cot\gamma+\cot\gamma\cot\alpha = 1$

(Proved)

Q.14 Express in the form of: $r\sin(\theta+\phi)$.

(i) $12\sin\theta+5\cos\theta$

Let $r\cos\phi = 12 \quad r\sin\phi = 5$

$r^2\cos^2\phi+r^2\sin^2\phi = 144+25$

$r^2(\sin^2\phi+\cos^2\phi) = 169$

$r^2 = 169 \Rightarrow r = 13$

Now $\frac{r\sin\phi}{r\cos\phi} = \frac{5}{12}$

$\Rightarrow \tan\phi = 5/12$

$\phi = \tan^{-1}(5/12)$

Thus

$12\sin\theta+5\cos\theta = r\sin\theta\cos\phi+r\cos\theta\sin\phi$

$= r[\sin\theta\cos\phi+\cos\theta\sin\phi]$

$= r\sin(\theta+\phi)$.

where $r=13 \quad \phi = \tan^{-1}(5/12)$.

(ii) $3\sin\theta-4\cos\theta$

let $r\cos\phi = 3 \quad r\sin\phi = -4$

$r^2\cos^2\phi+r^2\sin^2\phi = 9+16$

$r^2(\cos^2\phi+\sin^2\phi) = 25$

$r^2 = 25 \Rightarrow r = 5$

Now $\frac{r\sin\phi}{r\cos\phi} = \frac{-4}{3}$

$\tan\phi = -4/3$

$\Rightarrow \phi = \tan^{-1}(-4/3)$

Thus $3\sin\theta-4\cos\theta = r\sin\theta\cos\phi+r\cos\theta\sin\phi$

$= r[\sin\theta\cos\phi+\cos\theta\sin\phi]$

$= r\sin(\theta+\phi)$

where $r = 5$ and $\phi = \tan^{-1}(-4/3)$

(iii) $\sin\theta - \cos\theta$

let $r\cos\phi = 1$ $\quad r\sin\phi = -1$

$r^2\cos^2\phi + r^2\sin^2\phi = 1+1$

$r^2(\cos^2\phi + \sin^2\phi) = 2$

$r^2 = 2 \Rightarrow r = \sqrt{2}$

$\frac{r\sin\phi}{r\cos\phi} = \frac{-1}{1}$

$\tan\phi = -1 \Rightarrow \phi = \tan^{-1}(-1)$

$\phi = 135^\circ$

Thus $\sin\theta - \cos\theta = r\sin\theta\cos\phi + r\cos\theta\sin\phi$

$= r(\sin\theta\cos\phi + \cos\theta\sin\phi)$

$= r\sin(\theta+\phi)$

where $r = \sqrt{2}$ and $\phi = 135^\circ$

(iv) $5\sin\theta - 4\cos\theta$

let $r\cos\phi = 5$ $\quad r\sin\phi = -4$

$r^2\cos^2\phi + r^2\sin^2\phi = 25+16$

$r^2(\cos^2\phi + \sin^2\phi) = 41$

$r^2 = 41 \Rightarrow r = \sqrt{41}$

Now $\frac{r\sin\phi}{r\cos\phi} = \frac{-4}{5}$

$\tan\phi = -4/5 \Rightarrow \phi = \tan^{-1}(-4/5)$

Thus $5\sin\theta - 4\cos\theta = r\sin\theta\cos\phi + r\cos\theta\sin\phi$

$= r(\sin\theta\cos\phi + \cos\theta\sin\phi)$

$= r\sin(\theta+\phi)$

where $r = \sqrt{41}$ and $\phi = \tan^{-1}(-4/5)$

(v) $\sin\theta + \cos\theta$

let $r\cos\phi = 1$ $\quad r\sin\phi = 1$

$r^2\cos^2\phi + r^2\sin^2\phi = 1+1$

$r^2(\cos^2\phi + \sin^2\phi) = 2$

$r^2 = 2 \Rightarrow r = \sqrt{2}$

$\frac{r\sin\phi}{r\cos\phi} = \frac{1}{1} \Rightarrow \tan\phi = 1$

$\phi = \tan^{-1}(1) = 45^\circ$

Thus $\sin\theta + \cos\theta = r\sin\theta\cos\phi + r\cos\theta\sin\phi$

$= r(\sin\theta\cos\phi + \cos\theta\sin\phi)$

$= r(\sin(\theta+\phi))$

where $r = \sqrt{2}$ and $\phi = 45^\circ$

(vi) $3\sin\theta - 5\cos\theta$

let $r\cos\phi = 3$ $\quad r\sin\phi = -5$

$r^2\cos^2\phi + r^2\sin^2\phi = 9+25$

$r^2(\sin^2\phi + \cos^2\phi) = 34$

$r^2 = 34 \Rightarrow r = \sqrt{34}$

Now $\frac{r\sin\phi}{r\cos\phi} = \frac{-5}{3}$

$\tan\phi = -5/3 \Rightarrow \phi = \tan^{-1}(-5/3)$

Thus $3\sin\theta - 5\cos\theta = r\sin\theta\cos\theta + r\cos\theta\sin\phi$

$= r(\sin\theta\cos\phi + \cos\theta\sin\phi)$

$= r\sin(\theta+\phi)$

where $r = \sqrt{34}$ and $\phi = \tan^{-1}(-5/3)$

Double Angle Identities:-

$\because \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

Replace β by α

$\sin(\alpha+\alpha) = \sin\alpha\cos\alpha + \cos\alpha\sin\alpha$

$\sin 2\alpha = \sin\alpha\cos\alpha + \sin\alpha\cos\alpha$

$\boxed{\sin 2\alpha = 2\sin\alpha\cos\alpha}$

$\because \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

Replace β by α

$\cos(\alpha+\alpha) = \cos\alpha\cos\alpha - \sin\alpha\sin\alpha$

$\boxed{\cos 2\alpha = \cos^2\alpha - \sin^2\alpha}$

$\cos 2\alpha = \cos^2\alpha - (1-\cos^2\alpha)$

$= \cos^2\alpha - 1 + \cos^2\alpha$

$\boxed{\cos 2\alpha = 2\cos^2\alpha - 1}$

$\cos 2\alpha = 2(1-\sin^2\alpha) - 1$

$= 2 - 2\sin^2\alpha - 1$

$\boxed{\cos 2\alpha = 1 - 2\sin^2\alpha}$