Answers

Q.1:	i.	3	ii.	Zero	iii.	– 4	iv.	4
	v.	∞	vi.	∞				
Q.2:	i	$\frac{1}{2}$	ii.	1/4	iii.	4	iv.	$\frac{1}{2\sqrt{x}}$
	v.	- 2	vi	$-\frac{1}{2}$				
Q.3:	i	4	ii	$\frac{1}{4}$	iii	$\frac{1}{6}$	iv	$\frac{1}{2\sqrt{a}}$
Q.4:	i	8	ii	2	iii	$\frac{3}{2}$	iv	3
	v	$\frac{1}{2}$	vi	1	vii	$\frac{7}{2}$		

Q.5: i. $\frac{1}{4}$ ii. $-\frac{2}{9}$ iii. 3 iv. Zero v. $\frac{1}{3}$ vi. 0

1.13. Some Important Limits:

1.13.1 Prove that

$$\lim_{x \to 0} \frac{x^n - a^n}{x - a} = na^{n-1}$$

Proof:

If we substitute x = a, the function is of the form $\frac{0}{0}$ and is not defined.

Let x = a + h and limit h tends to zero

$$\frac{x^n - a^n}{x - a} = \frac{(a + h)^n - a^n}{(a + h) - a}$$

Using binomial expansion

$$= \frac{a^n + na^{n-1}h + \frac{n(n-1)}{2!}a^{n-2}h^2 + \ldots\ldots\ldots\ldots - a^n}{a + h - a}$$

$$= \frac{na^{n-1}h + \frac{n(n-1)}{2!}a^{n-2}h^2 + \ldots\ldots\ldots\ldots}{h}$$

$$= \frac{nh\left\{a^{n-1} + \frac{n-1}{2!}a^{n-2}h + \ldots\ldots\right\}}{h}$$

$$= n\left\{a^{n-1} + \frac{n-1}{2!}a^{n-2}h + \ldots\ldots\right\}$$

$$\lim_{x \to 0} \frac{x^n - a^n}{x - a} = \lim_{x \to 0} n\left\{a^{n-1} + \frac{n-1}{2}a^{n-2}h + \ldots\ldots\right\}$$

$$= n\{a^{n-1} + 0 + 0 + \ldots\ldots\ldots\ldots\}$$

$$\lim_{x \to 0} \frac{x^n - a^n}{x - a} = na^{n-1}$$

1.13.2 Prove that:

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$, where θ is in radian.

Proof:

Let r be the radius of a circle with center at 0 such that $m\angle AOB = \theta$.

Where θ is in radian

Draw the tangent at A. Produce OB such that It meets the tangent at T. Also join A and B as shown.

From the figure, in right angled triangle OAT.

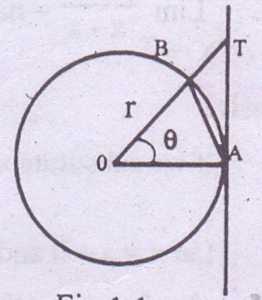

Fig.1.1

$$\frac{AT}{OA} = \tan\theta$$

$$\Rightarrow \quad AT = OA\tan\theta \qquad \because \qquad OA = r$$

$$AT = r\tan\theta$$

Also

$$\text{Area of triangle OAB} = \frac{1}{2}r^2\sin\theta$$

$$\text{Area of sector OAB} = \frac{1}{2}r^2\theta$$

$$\text{Area of triangle OAT} = \frac{1}{2}OA \times AT = \frac{1}{2}r^2\tan\theta$$

Now from the figure it is obvious that

Area of Δ AOB < Area of sector AOB < Area of Δ OAT

$$\frac{1}{2}r^2\sin\theta < \frac{1}{2}r^2\theta < \frac{1}{2}r^2\tan\theta$$

$\text{Sin}\,\theta < \theta < \tan\theta$

Dividing the inequality by sin θ.

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$

Or $$1 > \frac{\sin\theta}{\theta} > \cos\theta$$

When $\theta \to 0, \cos\theta \to 1$

$$1 > \lim_{\theta\to 0} \frac{\text{Sin}\,\theta}{\theta} > 1$$ (By using Sandwitch theorem)

Hence, $$\lim_{\theta\to 0} \frac{\text{Sin}\,\theta}{\theta} = 1$$

1.11.3 Prove that:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e = 2.7185$$

Solution:

Expand using binomial series

$$\left(1+\frac{1}{n}\right)^n = 1 + n\frac{1}{n} + \frac{n(n-1)}{2!}\ \frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}\ \frac{1}{n^3} + \ldots\ldots\infty$$

$$= 1 + 1 + \frac{1}{2!}\ n^2\left(1-\frac{1}{n}\right)\frac{1}{n^2} + \frac{1}{3!}\ n^3\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\frac{1}{n^3} + \ldots\ldots\infty$$

$$= 2 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \ldots\ldots\infty$$

Now

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty}\left[2+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\right]+\ldots\ldots\infty$$

When $n \to \infty$ all $\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots\ldots\ldots \to 0$

$$\therefore \quad \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 2 + \frac{1}{2!} + \frac{1}{3!} + \ldots\ldots\ldots$$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 2.712 = e$$

Example 1: **Evaluate the following limits**

i. $\lim_{x\to 0} \frac{1-\cos\theta}{\theta^2}$ ii. $\lim_{x\to 0} \frac{\sin px}{qx}$ iii $\lim_{x\to \pi/2} \frac{\sec x}{\tan x}$

iv. $\lim_{x\to 0} (1+3x)^{1/x}$

Solution:i

$$\lim_{x\to 0} \frac{1-\cos\theta}{\theta^2} = \lim_{\theta\to 0} \frac{1-\cos\theta}{\theta^2} \times \frac{1+\cos\theta}{1+\cos\theta}$$

$$= \lim_{\theta\to 0} \frac{1-\cos^2\theta}{\theta^2(1+\cos\theta)} = \lim_{\theta\to 0} \frac{\sin^2\theta}{\theta^2(1+\cos\theta)}$$

$$= \lim_{\theta\to 0} \left[\frac{\sin^2\theta}{\theta^2} \times \frac{1}{1+\cos\theta}\right]$$

$$= \left[\lim_{\theta\to 0} \frac{\sin\theta}{\theta}\right]^2 \left[\lim_{\theta\to 0} \frac{1}{1+\cos\theta}\right]$$

$$= [1]^2\left[\frac{1}{1+1}\right] = \frac{1}{2}$$

ii. $\lim_{x\to 0} \frac{\sin px}{qx}$

Solution:

$$\frac{\sin px}{qx} = \frac{\sin px}{px} \cdot \frac{p}{q}$$

$$\therefore \lim_{x\to 0} \frac{\sin px}{qx} = \frac{p}{q} \lim_{x\to 0} \frac{\sin px}{px}$$

$$= \frac{p}{q} 1 = \frac{p}{q}$$

iii $\lim_{x\to \pi/2} \frac{\sec x}{\tan x}$

Solution:

$$\lim_{x\to \pi/2} \frac{\sec x}{\tan x} = \lim_{x\to \pi/2} \frac{1/\cos x}{\sin x/\cos x} = \lim_{x\to \pi/2} \frac{1}{\sin x}$$

$$= \frac{1}{\sin \pi/2} = \frac{1}{1} = 1$$

iv. $\lim_{x\to 0} (1+3x)^{1/x}$

Solution:

$$\lim_{x\to 0}(1+3x)^{1/x} = \lim_{x\to 0}\left[(1+3x)^{1/3x}\right]^3$$

put $3x=\frac{1}{n}$, as $x\to 0, n\to\infty$

$$\therefore \lim_{x\to 0}\left[(1+3x)^{1/3x}\right]^3 = \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^n\right]^3 = e^3$$

Exercise 1.3

Q.1: Evaluate the following limits.

(i) $\lim_{\theta\to 0}\frac{\sin 7\theta}{\theta}$ (ii) $\lim_{x\to 0}\frac{\sin^2 x}{x^3}$

(iii) $\lim_{x\to 0}\frac{\tan x}{x}$ (iv) $\lim_{q\to 0}\frac{\tan pq}{q}$

(v) $\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta\sin\theta}$ (vi) $\lim_{x\to 0}\frac{1-\cos 2x}{x^2}$

(vii) $\lim_{x\to 0}\frac{1-\cos x}{\sin^2 x}$ (viii) $\lim_{x\to 0}\frac{1-\sin x}{\cos^2 x}$

(ix) $\lim_{\theta\to 0}\frac{1-\cos p\theta}{1-\cos q\theta}$ (x) $\lim_{\theta\to 0}\frac{\tan\theta-\sin\theta}{\sin^3\theta}$

(xi) $\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}$ (xii) $\lim_{x\to 0}\frac{\operatorname{cosec} x-\cot x}{x}$

Q.2: Evaluate the following limits:

(i). $\lim_{x\to 0}(1+x)^{1/x}$ (ii). $\lim_{x\to\infty}\left(1-\frac{1}{x}\right)^x$

(iii). $\lim_{x\to\infty}\left(\frac{x}{1+x}\right)^x$ (iv). $\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x$

(v). $\lim_{x\to 0}(1+2x)^{1/x}$

Q.3: Evaluate the following limits.

(i) $\lim_{h\to 0}\frac{(1+h)^{\frac{3}{2}}-1}{h}$ (ii) $\lim_{x\to 1}\frac{x^{\frac{3}{2}}-1}{x-1}$ (iii) $\lim_{x\to a}\frac{x^n-a^n}{x^m-a^m}$

Answers

Q.1:	i	7	ii.	∞	iii. 1		iv.	P	v.	$\frac{1}{2}$
	vi.	2	vii.	$\frac{1}{2}$	viii.	1	ix.	$\frac{p^2}{q^2}$	x	$\frac{1}{2}$
	xi.	$\frac{1}{2}$	xii.	$\frac{1}{2}$						
Q.2:	i.	e	ii.	1/e	iii.	1/e	iv.	e^2		
	v.	e^2								
Q.3:	i	$\frac{3}{2}$	ii	$\frac{3}{2}$	iii	$\frac{n}{m}a^{n-m}$				