Q.12: (i) Show that $\dfrac{e^x + 1}{e^x - 1}$ is an odd function of x.

(ii) show that $x \cdot \dfrac{a^x + 1}{a^x - 1}$ is an Even function of x

Q.13 : If $f(x) = \log \dfrac{1 - x}{1 + x}$ prove that $f(x) + f(y) = f\left(\dfrac{x + y}{1 + xy}\right)$

Answers

Q.1(i) 6 (ii) 35 **Q.2** $\dfrac{x^2}{1+4x^2}$ **Q.4** $a = -2$, $b = -6$

Q.11 (i) Neither (ii) Odd (iii) Even (iv) Odd

LIMITS

1.7 Introduction:

The concept of limit of a function is one of the fundamental ideas that distinguishes calculus from algebra and trigonometry. The concept of limit is the foundation of almost all the mathematical analysis. A good understanding of limit will help to explain many theories of calculus.

1.8 Value and Limit of a Function:

There are two ways of studying a function. One is to find the value of y, the dependent variable for various values of x, the independent variable by actual substitution in the functional relation between the two variables. Another method to study the behavior of the dependent variable in the neighborhood of various values of the independent variable. The former method is finding the ***value of the function*** and the later method, which gives us more detailed information about the function than the former, is finding the ***limit of the function.***

1.8.1 Limit of a Variable: $(x \to a)$

If a variable `x' approaches a constant `a' such that the absolute value of the difference x - a becomes less than 'ϵ', any +ve number, however small, then `a' is called the limit of x and is symbolically written as $x \to a$ and read as "x approaches a" or " x tends to a".

For Example:

Suppose that x has the values $a + \dfrac{1}{10}$, $a + \dfrac{1}{10^2}$, ..., $a + \dfrac{1}{10^n}$

Since the values $\dfrac{1}{10}, \dfrac{1}{10^2}, \ldots, \dfrac{1}{10^n} \to 0$ when $n \to \infty$

Hence $x \to a$

1.8.2 x approaches to ∞ or $-\infty$

If a variable go on increasing and the values are greater than any real number, however large, then we say that $x \to \infty$. Similarly when the values taken by x becomes smaller and smaller and are less than every negative number, then we say that $x \to -\infty$

1.8.3 Limit of a Function:

If a function f(x) is defined for all values of x in some interval and if x approaches a given number 'a' in that interval then f(x) may approaches some definite number 'ℓ' which is called the limit of f(x) as $x \to a$ and is symbolically written as:

$$\lim_{x \to a} f(x) = l$$

i.e. when x is very near to 'a' then f(x) is very near to ℓ.

1.9 Theorem on Limits of functions:

Let f(x) and g (x) are function and k be a real number, then

1. Constant $\lim_{x \to a} k = k$
2. Sum rule $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
3. Difference rule $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$
4. Product rule $\lim_{x \to a} [f(x)\, g(x)] = \lim_{x \to a} f(x) \lim_{x \to a} g(x)$
5. Constant multiple rule $\lim_{x \to a} kg(x) = k \lim_{x \to a} g(x)$
6. Quotient rule $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$; where $g(x) \neq 0$
7. Power rule If n is integer then $\lim_{x \to a} [f(x)]^n = \left(\lim_{x \to a} f(x)\right)^n$

1.10 Evaluation of limit of a function:

For evaluation of the limits of algebraic functions, there are two most important methods (a) direction method (b) indirect method

(a) Direct Method:

In cases, where values of the function does not assume any indeterminate form by putting the value of x directly to which it tends, in the function. The limit of the function is obtained.

Note : The indeterminate values of the function are of the form

$\frac{0}{0}, \frac{\infty}{\infty}, 0.\infty, \infty-\infty, 0^{\infty}, 1^{\infty}, \infty^{0}, \infty\times\infty$

Example 1: Evaluate $\lim_{x\to 3}(x^2+2x)$ (Direct Method)

Solution: Since the limit is not indeterminate form for x = 3, so put directly $\lim_{x\to 3}(x^2+2x) = 3^2+2(3) = 9+6 = 15$ Ans

(b) Indirect method:

Suppose as $x \to a$, the given limit is of the form 0/0 i.e., indeterminate

Then we use the following methods :

(1) **Factorization** (2) **Substitution** (3) **Rationalization**

(1) Factorization method :

1^{st} Step: Make factors of numerator and denominator.

2^{nd} Step: Cancel the common factor $x \to a$ (called disturbing factor) from numerator and denominator.

3^{rd} Step: Put x = a in the remaining function which is the required limit of the function.

Example 2: Evaluate $\lim_{x\to 1}\frac{x^2+x-2}{x^2-x}$

Solution: Given $\lim_{x\to 1}\frac{x^2+x-2}{x^2-x}$ is of ($\frac{0}{0}$ form) for x = 1.

Factorizing the numerator and denominator we get

$$\lim_{x\to 1}\frac{x^2+x-2}{x^2-x} = \lim_{x\to 1}\frac{(x-1)(x+2)}{x(x-1)}$$

$$= \lim_{x\to 1}\frac{x+2}{x}$$

$$= \frac{1+2}{1} = 3 \text{ Ans}$$

(2) Substitution Method(h- method):

Sometimes it becomes difficult to make the factors, then apply substitution, as

1^{st} Step: Put x = a + h in the given function.

2^{nd} Step: As $x \to a$, $h \to 0$ and simplify.

3rd Step: Cancel h (the disturbing factor) from numerator and denominator.
4th Step: Put h = 0 in the reduced function which gives the required limit.

Example 3: Evaluate $\lim_{x \to 3} \frac{x^3 - 4x^2 + 9}{x^2 - x - 6}$

Solution: Given limit is of indeterminate form $\frac{0}{0}$ for x = 3

Put $x = 3 + h$, when $x \to 3$, $h \to 0$

$$\therefore \quad \lim_{x \to 3} \frac{x^3 - 4x^2 + 9}{x^2 - x - 6} = \lim_{h \to 0} \frac{(3+h)^3 - 4(3+h)^2 + 9}{(3+h)^2 - (3+h) - 6}$$

$$= \lim_{h \to 0} \frac{27 + h^3 + 9h(3+h) - 4(9 + 6h + h^2) + 9}{9 + 6h + h^2 - 3 - h - 6}$$

$$= \lim_{h \to 0} \frac{27 + h^3 + 27h + 9h^2 - 36 - 24h - 4h^2 + 9}{h^2 + 5h}$$

$$= \lim_{h \to 0} \frac{h^3 + 5h^2 + 3h}{h^2 + 5h}$$

$$= \lim_{h \to 0} \frac{h(h^2 + 5h + 3)}{h(h+5)}, \text{ cancel } h$$

$$= \lim_{h \to 0} \frac{h^2 + 5h + 3}{h + 5} = \frac{0 + 0 + 3}{0 + 5} \text{ (putting } h = 0\text{)}$$

$$= \frac{3}{5} \text{ Ans}$$

(3) Method of Rationalization:

In the case of irrational functions, if limit of f(x) is of the form $\frac{0}{0}$ i.e., indeterminate, proceed as under:

1st Step: Rationalize the numerator or denominator as the case may be.
2nd Step: Cancel the common factor (disturbing factor) from numerator and denominator i.e., $x - a$ if $x \to a$
3rd Step: Put x = a in the reduced function, which is the required value of the given limit.

Example 4: Evaluate the limits:

(i) $\lim\limits_{x \to 0} \dfrac{\sqrt{2+x}-\sqrt{2}}{x}$ (ii) $\lim\limits_{x \to 0} \dfrac{\sqrt{1-x}-\sqrt{1+x}}{x}$

Solution:(i) Given limit $\lim\limits_{x \to 0} \dfrac{\sqrt{2+x}-\sqrt{2}}{x}$ is of ($\dfrac{0}{0}$ form) for x = 0.

Rationalize the numerator.

$$\therefore \quad \lim_{x \to 0} \frac{\sqrt{2+x}-\sqrt{2}}{x} = \lim_{x \to 0} \frac{\sqrt{2+x}-\sqrt{2}}{x} \cdot \frac{\sqrt{2+x}+\sqrt{2}}{\sqrt{2+x}+\sqrt{x}}$$

$$= \lim_{x \to 0} \frac{2+x-2}{x(\sqrt{2+x}+\sqrt{x})}$$

$$= \lim_{x \to 0} \frac{1}{\sqrt{2+x}+\sqrt{2}} = \frac{1}{\sqrt{2+0}+\sqrt{2}}$$

$$\therefore \quad \lim_{x \to 0} \frac{\sqrt{2+x}+\sqrt{2}}{x} = \frac{1}{2\sqrt{2}} \text{ Ans}$$

Solution:(ii) Given $\lim\limits_{x \to 0} \dfrac{\sqrt{1-x}-\sqrt{1+x}}{x}$; $\dfrac{0}{0}$ form for x = 0

Rationalize the numerator

$$\therefore \quad \lim_{x \to 0} \frac{\sqrt{1-x}-\sqrt{1+x}}{x} = \lim_{x \to 0} \frac{\sqrt{1-x}-\sqrt{1+x}}{x} \times \frac{\sqrt{1-x}+\sqrt{1+x}}{\sqrt{1-x}+\sqrt{1+x}}$$

$$= \lim_{x \to 0} \frac{1-x-1-x}{x(\sqrt{1-x}+\sqrt{1+x})}$$

$$= \lim_{x \to 0} \frac{-2x}{x(\sqrt{1-x}+\sqrt{1+x})}$$

$$= \lim_{x \to 0} \frac{-2}{\sqrt{1-x}+\sqrt{1+x}}$$

$$= \frac{-2}{\sqrt{1-0}+\sqrt{1+0}} = \frac{-2}{1+1} = \frac{-2}{2}$$

$$\therefore \quad \lim_{x \to 0} \frac{\sqrt{1-x}-\sqrt{1+x}}{x} = -1 \text{ Ans}$$

1.11 Limit when x tends to infinity

Consider a function f(x) defined for large positive (or negative) value of x as increases indefinitely in the positive (or negative) direction. If the value of f(x) approaches a number 'b' then the limit of f(x) as x increases (or decreases) indefinitely is equal to b, denoted as

$$\lim_{x \to +\infty} f(x) = b \quad \text{or} \quad \lim_{x \to -\infty} f(x) = b$$

Note : $\lim_{x\to+\infty} \frac{1}{x} = 0, \lim_{x\to-\infty} \frac{1}{x} = 0, \lim_{x\to\infty} \frac{1}{x} = 0$

1.12 Evaluation of Limit when x tends to ∞ :

When $x \to \infty$ and the limit takes the form $\frac{\infty}{\infty}$, we can use two methods.

Method - I:

Divide numerator and denominator by the highest power of x

Method – II:

Put $x = \frac{1}{y}$, then when $x \to \infty, y \to 0$

Example 5:

Method – I : Find $\lim_{x\to\infty} \frac{2x^2 - 3x + 7}{3x^2 + 5x - 1}$

Divide numerator and Denominator by highest power of x i.e., x^2

$$= \lim_{x\to\infty} \frac{2 - \frac{3}{x} + \frac{7}{x^2}}{3 + \frac{5}{x} + \frac{1}{x^2}}$$

$$= \frac{\lim_{x\to\infty}\left(2 - \frac{3}{x} + \frac{7}{x^2}\right)}{\lim_{x\to\infty}\left(3 - \frac{5}{x} + \frac{7}{x^2}\right)} = \frac{2 - 0 + 0}{3 + 0 - 0} = \frac{2}{3}$$ Ans.

Method – II:

Put $x = \frac{1}{y}$

When $x \to \infty, \quad y \to 0$

So,

$$\lim_{x\to\infty} \frac{2x^2 - 3x + 7}{3x^2 + 5x - 1} = \lim_{y\to 0} \frac{\frac{2}{y^2} - \frac{3}{y} + 7}{\frac{3}{y^2} + \frac{5}{y} - 1}$$

$$= \lim_{y\to 0} \frac{2 - 3y + 7y^2}{3 + 5y - y^2}$$

$$= \frac{2 - 3(0) + 7(0)}{3 + 5(0) - 0} = \frac{2}{3} \quad \text{Ans.}$$

Example 6:

Evaluate $\lim_{x\to\infty} \frac{\sqrt{x^2 - 1}}{2x + 1}$

Solution:

We first divide numerator and denominator by x and get

$$\lim_{x\to\infty} \frac{\sqrt{x^2 - 1}}{2x + 1} = \lim_{x\to\infty} \frac{\sqrt{1 - \frac{1}{x^2}}}{2 + \frac{1}{x}}$$

$$= \frac{\lim_{x\to\infty} \sqrt{1 - \frac{1}{x^2}}}{\lim_{x\to\infty} \left(2 + \frac{1}{x}\right)} = = \frac{\sqrt{1-0}}{2+0} = \frac{1}{2} \quad \text{Ans.}$$

Exercise 1.2

Q.1: Evaluate the following limits

(i) $\lim_{x\to 1} (x^3 - 3x + 5)$ (ii) $\lim_{x\to 1} \frac{\sqrt{x - 1}}{\sqrt{x + 1}}$ (iii) $\lim_{x\to -2} \frac{x^2}{x + 1}$

(iv) $\lim_{x\to 3} \sqrt{25 - x^2}$ (v) $\lim_{x\to 0} \frac{4x+5}{x}$ (vi) $\lim_{x\to 0} \frac{2x^2 + 5x + 7}{x}$

Q.2: Evaluate the following limit

(i) $\lim\limits_{x \to 0} \dfrac{\sqrt{1+x}-1}{x}$ (ii) $\lim\limits_{x \to 0} \dfrac{\sqrt{4+x}-2}{x}$

(iii) $\lim\limits_{x \to 0} \dfrac{x}{\sqrt{4+x}-2}$ (iv) $\lim\limits_{h \to 0} \dfrac{\sqrt{x+h}-\sqrt{x}}{h}$

(v) $\lim\limits_{h \to 0} \dfrac{(1-h)^2-1}{h}$ (vi) $\lim\limits_{h \to 0} \left[\dfrac{1}{h}\left(\dfrac{1}{\sqrt{1-h}}-1\right)\right]$

Q.3 : Evaluate the following limits.

(i) $\lim\limits_{x \to 1} \dfrac{x-1}{\sqrt{x+3}-2}$ (ii) $\lim\limits_{x \to 1} \dfrac{\sqrt{x}-3}{x-9}$

(vi) $\lim\limits_{x \to 2} \dfrac{\sqrt{x+7}-3}{x-2}$ (iv) $\lim\limits_{x \to a} \dfrac{\sqrt{x}-\sqrt{a}}{x-a}$

Q.4: Evaluate the following limits.

(i) $\lim\limits_{x \to 4} \dfrac{x^2-16}{x-4}$ (ii) $\lim\limits_{x \to 1} \dfrac{x^2-1}{x-1}$

(iii) $\lim\limits_{x \to 1} \dfrac{x^3-1}{x^2-1}$ (iv) $\lim\limits_{x \to 2} \dfrac{x^2-x-2}{x^2-3x+2}$

(v) $\lim\limits_{x \to 0} \dfrac{x^2-3x+2}{x^2-4x+4}$ (vi) $\lim\limits_{x \to 3} \dfrac{x^2-5x+6}{x-3}$

(vii) $\lim\limits_{x \to -3} \dfrac{x^2-x-12}{x^2+4x+3}$

Q.5: Calculate the following limits.

(i). $\lim\limits_{x \to \infty} \dfrac{x^3+8x^2+6}{4x^3+7x^2-5x}$ (ii). $\lim\limits_{x \to \infty} \dfrac{2x^3-3x+7}{6x-7x^2-9x^3}$

(iii). $\lim\limits_{x \to \infty} \dfrac{3x^2-5x+1}{x^2+x+2}$ (iv). $\lim\limits_{x \to \infty} \dfrac{x^2+1}{2x^3-x}$

(v). $\lim\limits_{x \to \infty} \dfrac{\sqrt{x^2-1}}{3x+1}$ (vi). $\lim\limits_{x \to \infty} \left(\sqrt{x^2+1}-x\right)$

Answers

Q.1: i. 3 ii. Zero iii. – 4 iv. 4
v. ∞ vi. ∞

Q.2: i $\frac{1}{2}$ ii. 1/4 iii. 4 iv. $\frac{1}{2\sqrt{x}}$
v. - 2 vi $-\frac{1}{2}$

Q.3: i 4 ii $\frac{1}{4}$ iii $\frac{1}{6}$ iv $\frac{1}{2\sqrt{a}}$

Q.4: i 8 ii 2 iii $\frac{3}{2}$ iv 3
v $\frac{1}{2}$ vi 1 vii $\frac{7}{2}$

Q.5: i. $\frac{1}{4}$ ii. $-\frac{2}{9}$ iii. 3 iv. Zero v. $\frac{1}{3}$ vi. 0

1.13. Some Important Limits:

1.13.1 Prove that

$$\lim_{x \to 0} \frac{x^n - a^n}{x - a} = na^{n-1}$$

Proof:

If we substitute x = a, the function is of the form $\frac{0}{0}$ and is not defined.

Let x = a + h and limit h tends to zero

$$\frac{x^n - a^n}{x - a} = \frac{(a+h)^n - a^n}{(a+h) - a}$$

Using binomial expansion

$$= \frac{a^n + na^{n-1}h + \frac{n(n-1)}{2!}a^{n-2}h^2 + \ldots\ldots\ldots\ldots\ldots -a^n}{a + h - a}$$